

Study of Charged Non-relativistic Fluids using Light Cone Reduction

Akash Jain, Department of Physics, IISER Bhopal
under the supervision of Dr. Suvankar Dutta (IISER Bhopal) and Dr. Nabamita Banerjee (IACS Kolkata)

Contact Information:

Akash Jain
BS-MS Final Year,
Department of Physics, IISER Bhopal
Email: ajainphysics@gmail.com



Introduction and Summary

- We aim to study properties of a $(d, 1)$ -dim charged non-relativistic fluid.
- We consider a generic charged $(d + 1, 1)$ -dim relativistic fluid, and undergo light cone reduction.
- We also add all possible parity-odd terms to our theory, in first derivative order, specific to $d = 2$.
- Light cone reduction of the above system gives a $(d, 1)$ -dim non-relativistic theory. We determine the parameters of non-relativistic fluid in terms of the parameters of relativistic fluid.
- We find that parity-even transport coefficients have same constraints in both relativistic and non-relativistic regime.
- Parity-odd transport coefficients (for $d = 2$) however follow different constraints in both cases.
- In particular, parity odd coefficients in non-relativistic theory can only survive for incompressible flow in electromagnetic background with constant magnetic field.

Relativistic fluid dynamics

Thermodynamic relations:

$$E + P = TS + M_I Q_I, \quad dE = TdS + M_I dQ_I. \quad (1)$$

Constitutive equations:

$$\nabla_\mu T^{\mu\nu} = F_I^{\nu\lambda} J_{I\lambda}, \quad \nabla_\mu J_I^\mu = \{C_{IJK} E_J^\mu B_{K\mu}\}, \quad (2)$$

where energy-momentum tensor is:

$$T^{\mu\nu} = (E + P)u^\mu u^\nu + P g^{\mu\nu} - 2\eta\tau^{\mu\nu} - \zeta P^{\mu\nu} \nabla_\mu u^\mu, \quad (3)$$

$$\tau^{\mu\nu} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} \left[\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d+1} g_{\alpha\beta} \theta \right], \quad (4)$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \quad u^\mu u_\mu = -1, \quad (5)$$

charge current is

$$J_I^\mu = Q_I u^\mu + \Upsilon_I^\mu \quad (6)$$

$$\Upsilon_I^\mu = -\varrho_{IJ} P^{\mu\nu} \nabla_\nu \left(\frac{M_J}{T} \right) + \lambda_{IJ} E_J^\mu - \gamma_I P^{\mu\nu} \nabla_\nu T + \{ \tilde{\mathcal{U}}_I l^\mu + \tilde{\mathcal{U}}_{IJ} B_J^\mu \}, \quad (7)$$

$$l^\mu = \epsilon^{\mu\alpha\beta\gamma} u_\alpha \nabla_\beta u_\gamma, \quad E_I^\mu = F_I^{\mu\nu} u_\nu, \quad B_I^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{I\alpha\beta}, \quad (8)$$

and background fields are defined as:

$$F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu. \quad (9)$$

Entropy current:

$$\nabla_\mu J_S^\mu \geq 0, \quad J_S^\mu = S u^\mu - \frac{M_I}{T} \Upsilon_I^\mu + \{ D l^\mu + \tilde{D}_I B_I^\mu \}. \quad (10)$$

It implies constraints on transport coefficients as:

$$\gamma_I = 0, \quad \eta \geq 0, \quad \zeta \geq 0, \quad (11)$$

$$\lambda_{IJ} = \frac{1}{T} \varrho_{IJ}, \quad \varrho_{IJ} \text{ matrix is positive definite}, \quad (12)$$

and $D, \tilde{D}_I, \mathcal{U}_I, \tilde{\mathcal{U}}_{IJ}$ are related to C_{IJK} .

Non-relativistic fluid dynamics

Thermodynamic relations:

$$2(\epsilon + p) = st + q_I \mu_I, \quad 2dp + d\epsilon = sdt + q_I d\mu_I + \frac{\epsilon + p}{\rho} d\rho. \quad (13)$$

Constitutive Equations:

$$\partial_t(\rho v^j) + \partial_i(t^{ij}) = q_I \epsilon_I^j - j_{Ii} \beta_I^{ij}, \quad \partial_t \left(\epsilon + \frac{1}{2} \rho v^2 \right) + \partial_i j^i = j_I^i \epsilon_{Ii}, \quad (14)$$

$$\partial_t \rho + \partial_i(\rho v^i) = 0, \quad \partial_t q_I + \partial_i j_I^i = 0, \quad (15)$$

where stress-energy tensor is given by:

$$t^{ij} = \rho v^i v^j + p g^{ij} - n \sigma^{ij} - z \delta^{ij} \partial_k v^k, \quad (16)$$

$$\sigma^{ij} = \partial^i v^j + \partial^j v^i - \delta^{ij} \frac{2}{d} \partial_k v^k, \quad (17)$$

energy current is given by:

$$j^i = \left(\epsilon + p + \frac{1}{2} \rho v^2 \right) v^i - n \sigma^{ij} v^j - z \partial_k v^k v^i - \kappa \partial^i t - \kappa_I \nabla^i \left(\frac{\mu_I}{t} \right) + \frac{\kappa_I}{t} (\epsilon_I^i - v_j \beta_I^{ji}), \quad (18)$$

charge current is given by:

$$j_I^i = q_I v^i + v_I^i, \quad (19)$$

and background fields are:

$$\epsilon_I^i = -\partial^i \phi_I - \partial_t a_I^i, \quad \beta_I^{ij} = \partial^i a_I^j - \partial^j a_I^i. \quad (20)$$

Entropy Current

$$\partial_t s + \partial_i j_S^i \geq 0, \quad j_S^i = s v^i - \frac{\mu_I}{t} v_I^i. \quad (21)$$

Light Cone Reduction

- Light Cone Reduction is a formalism which relates a $(d + 1, 1)$ -dim relativistic theory (Poincaré invariance) to a $(d, 1)$ -dim non-relativistic theory (Galilean/Schrödinger invariance).
- Consider a $(d + 1, 1)$ -dim relativistic fluid theory in light cone coordinates $\{x^+, x^-, \mathbf{x}\}$.
- The relativistic theory is then reduced along x^- direction, i.e. all the parameters of the fluid are considered to be independent of x^- .
- Identifying x^+ with time, we will get a $(d, 1)$ -dim non-relativistic theory through this mechanism.
- The constitutive relations of the non-relativistic fluid theory can be reached from the relativistic ones if we identify:

$$T^{++} = \rho, \quad T^{i+} = \rho v^i, \quad (22)$$

$$T^{+-} = \epsilon + \frac{1}{2} \rho v^2, \quad T^{i-} = j^i, \quad T^{ij} = t^{ij}, \quad (23)$$

$$J_I^+ = q_I, \quad J_I^i = j_I^i, \quad (24)$$

$$A_I^- = \phi_I, \quad A_I^i = a_I^i, \quad A^+ = \text{constant} \quad (25)$$

Parameters of non-relativistic theory

Using the light cone identification we can compute the non-relativistic parameters in terms of relativistic:

$$\rho = (E + P)(u^+)^2 + (u^+)^2 (\eta \mathbf{Z} - \zeta \theta), \quad v^i = \frac{u^i}{u^+} - \frac{\eta}{\rho} \mathbf{Y}^i, \quad (26)$$

$$p = P - \left(\frac{\eta}{d} \mathbf{Z} - \zeta \frac{u^\alpha \nabla_\alpha P}{E + P} \right), \quad \epsilon = \frac{1}{2} (E - P) - \frac{1}{2} (\eta \mathbf{Z} - \zeta \theta), \quad (27)$$

$$t = \frac{T}{u^+} + \mathcal{O}(1), \quad \mu_I = \frac{M_I}{u^+} + \mathcal{O}(1), \quad (28)$$

$$q_I = J_I^+ = u^+ Q_I - u^+ \left[\varrho_{IJ} u^\nu \nabla_\nu \left(\frac{M_J}{t} \right) + \gamma_I u^\nu \nabla_\nu T + \{ \mathcal{U}_I \epsilon^{ij} u^+ \nabla_i v_j + \tilde{\mathcal{U}}_{IJ} \epsilon_{ij} \beta_I^{jj} \} \right]. \quad (29)$$

$$j_Q^i = q_I v^i - \xi \nabla_k t - r \nabla_k \left(\frac{\mu}{t} \right) - m \nabla_k p + l (\epsilon_j - v^k \beta_{kj}), \quad (30)$$

Transport coefficients:

$$n = \eta u^+, \quad z = \zeta u^+, \quad \kappa = 2n \frac{\epsilon + p}{t\rho}, \quad \kappa_I = n \frac{q_I t}{\rho}, \quad (31)$$

$$\omega_I = \mathcal{U}_I (u^+)^2, \quad \tilde{\omega}_{IJ} = \tilde{\mathcal{U}}_{IJ} u^+, \quad (32)$$

$$m_I^{ik} = m_I g^{ik} + \left\{ \frac{2\omega_I}{\rho} \epsilon^{ik} \right\}, \quad m_I = \frac{t\gamma_I u^+}{2(\epsilon + p)}, \quad (33)$$

$$r_{IJ}^{ik} = r_{IJ} g^{ik} - \left\{ \frac{\omega_{IJ} q_J t}{\rho} \epsilon^{ik} \right\}, \quad r_{IJ} = \left[\varrho_{IJ} - m_I q_J t + \kappa t^2 \frac{q_I q_J}{4(\epsilon + p)^2} \right], \quad (34)$$

$$l_{IJ}^{ij} = l_{IJ} g^{ij} + \left\{ \frac{\omega_{IJ} q_J}{\rho} \epsilon^{ij} - 2\tilde{\omega}_{IJ} \epsilon^{ij} \right\}, \quad l_{IJ} = \left[\lambda_{IJ} u^+ + \kappa t \frac{q_I q_J}{4(\epsilon + p)^2} \right], \quad (35)$$

$$\xi_I^{ik} = \xi_I g^{ik} - \left\{ \frac{2\omega_I (\epsilon + p)}{\rho t} \epsilon^{ik} \right\}, \quad \xi_I = \frac{n q_I}{\rho t}. \quad (36)$$

Entropy positivity of non-relativistic fluid

- Once equipped with the form of j^i , we can write the entropy positivity equation for non-relativistic fluids, which will give the same constraints as for relativistic case for the parity-even transport coefficients.

$$m_I = 0, \quad n \geq 0, \quad z \geq 0, \quad (37)$$

$$l_{IJ} = \frac{1}{t} r_{IJ}, \quad \left[r_{IJ} - \frac{nt q_I q_J}{2\rho(\epsilon + p)} \right] \text{ matrix is positive definite.} \quad (38)$$

- For fluids in 3 or more spatial dimensions, parity-odd terms do not appear at first derivative order, and thus we have a complete description of such systems.
- However for fluids in 2 spatial dimensions, positive-definiteness of parity-odd terms require us to add extra corrections to the entropy current.
- We found that this can be consistently done only for incompressible flow in an electromagnetic background with constant magnetic field.
- For any other scenarios however, the parity-odd transport coefficients must vanish.