Non-Relativistic Fluids through Light Cone Reduction

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Based on: Nabamita Banerjee, Suvankar Dutta and **Akash Jain**, *'On equilibrium partition function for non-relativistic fluid'*, [arXiv:1505.05677]

Abstract

We construct an equilibrium partition function and entropy current for a non-relativistic fluid and use it to constrain the dynamics of the system. The construction is based on light cone reduction, which is known to reduce the Poincaré symmetry to Galilean in one lower dimension. We modify the constitutive relations of relativistic fluid, and find that its symmetry broken phase – 'null fluid' is equivalent to the non-relativistic fluid. In particular, their symmetries, thermodynamics, constitutive relations, and equilibrium partition function match exactly to all orders in derivative expansion.

Introduction

It is known that Newton-Cartan geometry with Galilean symmetry (non-relativistic) follows from null-reduction of a relativistic geometry in one higher dimension. The idea behind this is that the Poincaré algebra in 5-dim has a 4-dim Galilean subalgebra embedded into it. More precisely we define *'null background'* – a 5 dimensional spacetime $\mathcal{M}$ equipped with a metric G_{MN} which admits a null Killing vector field V^M such that $\hat\nabla_M V^N = 0$. Null backgrounds have Galilean symmetry, and upon compactification along the V direction gives rise to 4 dimensional non-relativistic spacetime. We aim to use this mechanism to study non-relativistic hydrodynamics.

We want to study dynamics of a fluid on these null backgrounds, which we call *null fluid*. It differs the usual relativistic fluid since the constitutive relations of null fluid also depend on the additional background data – the null isometry V. This gives rise to additional tensor structure and transport coefficients in constitutive relations. In particular we will have an additional 'density' (which will be identified as mass density) in ideal order constitutive relations, and hence the thermodynamics of null fluid is different from usual fluids. The main result of this work is that null fluid is found to be in exact equivalence with non-relativistic fluid in one lower dimension.

Null Fluid

A physical theory on null background is described by (log of) a partition function which is a functional of metric, $W = W[G_{MN}]$. We parametrize its variation using an energy-momentum tensor,

$$\delta W = \frac{1}{2}\int \left\{\mathrm{d}x^M\right\} \sqrt{-G}\, \mathcal{T}^{MN} \delta G_{MN}. \tag{1}$$

Demanding invariance of partition function under diffeomorphisms would imply energy-momentum conservation law,

$$\hat\nabla_M \mathcal{T}^{MN} = 0. \tag{2}$$

We have 5 conservation equations, so any system with 5 parameters would be exactly solvable. We choose to describe our system by a fluid with parameters,

Null Velocity: u^M, Temperature: ϑ, Chemical Potential: ϖ,

such that $u^M u_M = 0$ and $u^M V_M = -1$. Due to dissipation, a fluid is not described by a partition function, but is rather characterized by most generic form of $\mathcal{T}^{MN}$ (upon fixing a hydrodynamic frame), known as *constitutive relations*,

$$\mathcal{T}^{MN} = R u^M u^N + 2E u^{(M} V^{N)} + P P^{MN}_{(u)} + 2\mathcal{E}^{(M} V^{N)} + \Pi^{MN}, \tag{3}$$

where we have defined thermodynamics and projection operator on null backgrounds,

$$\mathrm{d}P = \frac{E+P}{\vartheta}\mathrm{d}\vartheta + \vartheta R \mathrm{d}\varpi, \quad E + P = S\vartheta + \vartheta R\varpi, \quad P^{MN}_{(u)} = 2u^{(M}V^{N)} + G^{MN}. \tag{4}$$

$\mathcal{E}^M, \Pi^{MN}$ contain derivative corrections transverse to u^M and V^M. To leading derivative order, most generic form of (derivative corrections to) constitutive relations is given as,

$$\Pi^{MN} = -\eta\sigma^{MN} - \zeta P^{MN}_{(u)}\Theta, \qquad \mathcal{E}^M = P^{MN}_{(u)}\left(\kappa\partial_N\vartheta + \lambda\partial_N\varpi\right) + \tilde\omega\epsilon^{MNRST} V_N u_R \partial_S u_T,$$

$$\sigma^{MN} = P^{MR}_{(u)} P^{NS}_{(u)} \left(\hat\nabla_R u_S + \hat\nabla_S u_R - \frac{2}{3}P_{(u)RS}\nabla_M u^M\right), \qquad \Theta = \nabla_M u^M. \tag{5}$$

η, ζ, κ, λ, $\tilde\omega$ are arbitrary functions of ϑ, ϖ known as *transport coefficients*. They are constrained by certain physical requirements, like existence of an entropy current and equilibrium.

Unlike usual relativistic fluids, constitutive relations of null fluids depend on the Killing vector field V. This allows for additional tensor structure which must be considered to get most generic non-relativistic fluid via reduction. In a previous work we have noted that if we fail to include these terms, the non-relativistic fluid we get is highly restricted.

Entropy Current: Second law of thermodynamics requires that there must exist an entropy current $\mathcal{J}^M_s$ whose divergence is positive semi-definite $\hat\nabla_M \mathcal{J}^M_s \geq 0$. Through some generality arguments one can find such entropy current,

$$\mathcal{J}^M_s = S u^M + \frac{1}{\vartheta}\mathcal{E}^M + C_1\vartheta\epsilon^{MNRST} V_N u_R \partial_S u_T, \tag{6}$$

which implies constraints on transport coefficients,

$$\eta, \zeta \geq 0, \qquad \kappa \leq 0, \qquad \lambda = 0, \qquad \tilde\omega = 2\vartheta^2\left(C_1 + \frac{S}{R}C_2\right). \tag{7}$$

Here C_1, C_2 are some arbitrary constants.

Equilibrium Partition Function: If allowed to settle for long enough, all physical systems must admit an equilibrium configuration. More precisely, there must exist a partition function W^{eqb}, from which constitutive relations of fluid should be derived in equilibrium configuration. In general this requirement only gives equality type constraints, and are a subset of the entropy current constraints. We define the system to be in equilibrium if it admits a timelike Killing field K^M. We pick up a basis $x^M = \{x^-, x^+, x^i\}$, and use the diffeomorphism invariance to choose $V = \partial/\partial x^-$ and $K = \partial/\partial x^+$. Metric can then be decomposed as,

$$\mathrm{d}s^2 = -2e^{-\Phi}(\mathrm{d}x^+ + a_i\mathrm{d}x^i)(\mathrm{d}x^- - \mathcal{B}_+\mathrm{d}x^+ - \mathcal{B}_i\mathrm{d}x^i) + g_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{8}$$

Restricted to this choice of basis, we have independent background data,

Scalars: Φ, $\mathcal{B}_+$, Vector Gauge Fields: a_i, $B_i = \mathcal{B}_i - a_i\mathcal{B}_+$, Metric: g_{ij}. (9)

At equilibrium, the partition function is a gauge and 3 dimensional diffeomorphism invariant scalar functional of these background fields $W^{eqb} = W^{eqb}[\Phi, a_i, \mathcal{B}_+, B_i, g_{ij}]$. The variation of partition function eqn. (1) will then decompose as,

$$\delta W^{eqb} = \int\left\{\mathrm{d}x^i\right\}\sqrt{g}\frac{1}{\vartheta_o}\left[(\mathcal{T}_{+-} + \mathcal{T}_{--}\mathcal{B}_+)\frac{e^{\Phi}}{\vartheta_o}\delta\vartheta_o + \mathcal{T}^i_+\delta a_i + \frac{1}{2}\mathcal{T}^{ij}\delta g_{ij} + \vartheta_o\mathcal{T}_{--}\delta\varpi_o - \mathcal{T}^i_-\delta B_i\right]. \tag{10}$$

which only depends on x^i. Here we have defined $\vartheta_o = \tilde\vartheta e^{\Phi}$ and $\varpi_o = \mathcal{B}_+/\tilde\vartheta$. $\tilde\vartheta = 1/(\tilde\beta\tilde R)$ where $\tilde\beta$ is the radius of the euclidean time x^+ and $\tilde R$ is the radius of compactified x^-. To leading derivative order, its most generic form is given as,

$$W^{eqb} = \int\left\{\mathrm{d}x^i\right\}\sqrt{g}\left[\frac{1}{\vartheta_o}P_o - \epsilon^{ijk}\left(2C_1\tilde\vartheta a_i\partial_j B_k + C_2 B_i\partial_j B_k\right)\right]. \tag{11}$$

P_o is an arbitrary function of ϑ_o, ϖ_o which is equilibrium value of thermodynamic pressure P. The parity odd part are Chern-Simons terms and are only gauge invariant upto some boundary terms, therefore their integral in gauge invariant on a manifold without boundary. One can check that on varying this partition function, we get both the equality type constraints which we got from entropy current.

Light Cone Reduction

Now, we consider a *null background* and compactify it along the null direction V. It is known that the theories on this background have non-rel invariance. To make the connection more rigorous, we pick up an arbitrary vector field $T(\neq V)$, and use it to define a unique foliation of $\mathcal{M} = S^1_V \times R^1_T \times \mathcal{M}^T$,

$$\mathcal{M}^T := \left\{v^N : v^M V_M = v^M T_M = 0\right\}. \tag{12}$$

Since the theories on null background do not depend on the choice of T, they must be invariant under arbitrary redefinition $T^M \to T^M + \zeta^M$, where $\zeta^M \in \mathcal{M}$. The choice of T amounts to choosing a reference frame in the non-relativistic theory, and T redefinition is equivalent to Galilean Boosts.

For each T, we can define another null vector field $\bar V = \frac{1}{\alpha}\left(T^M + \frac{\omega}{2\alpha}V^M\right)$, where $\omega = T_M T^M$, $\alpha = -T^M V_M$; and a projection operator $P^{MN} = G^{MN} + 2\bar V^{(M}V^{N)}$. Check that $V^M\bar V_M = -1$. Using V^M and $\bar V^M$, we can decompose the energy-momentum tensor $\mathcal{T}^{MN}$ as,

$$\mathcal{T}^{MN} = \hat\rho\bar V^M\bar V^N + 2\hat\epsilon_{tot}\bar V^{(M}V^{N)} + 2j^{(M}_\rho\bar V^{N)} + 2j^{(M}_\epsilon V^{N)} + t^{MN}, \tag{13}$$

where,

Mass Density: $\hat\rho = \mathcal{T}^{RS}V_R V_S$, Mass Current: $j^M_\rho = -\mathcal{T}^{RS}P^M_R V_S$,

Energy Density: $\hat\epsilon_{tot} = \mathcal{T}^{RS}V_R\bar V_S$, Energy Current: $j^M_\epsilon = -\mathcal{T}^{RS}P^M_R\bar V_S$,

Stress-Energy Tensor: $t^{MN} = \mathcal{T}^{RS}P^M_R P^N_S$.

The relativistic energy-momentum conservation laws eqn. (2) will decompose into,

$$\begin{aligned}
&\text{Mass Conservation:} && \hat\nabla_M\left(\hat\rho\bar V^M + j^M_\rho\right) = 0,\\
&\text{Energy Conservation:} && \hat\nabla_M\left(\hat\epsilon_{tot}\bar V^M + j^M_\epsilon\right) = -\left(\bar V^M\hat p^N + t^{MN}\right)\hat\nabla_M\bar V_N,\\
&\text{Momentum Conservation:} && P_{NR}\hat\nabla_M\left(\bar V^M j^R_\rho + t^{MR}\right) = -\left(\hat\rho\bar V^M + j^M_\rho\right)\hat\nabla_M\bar V_N.
\end{aligned} \tag{14}$$

Non-Relativistic Fluids

On choosing a basis $x^M = \{x^-, x^+, x^i\}$ such that $V = \partial/\partial x^-$ and $T = \partial/\partial x^+$, we can get the non-relativistic fluid from null fluid. These will satisfy the conservation equations (on time-independent $(\partial_+ g_{ij} = 0)$ backgrounds with flat time $(a_i = \Phi = 0)$ for simplicity),

$$\partial_+ R + \nabla_i\left(Rv^i\right) = 0, \qquad \partial_+\left(E + \frac{1}{2}Rv^i v_i\right) + \nabla_i j^i_\epsilon = j^i_\rho\mathfrak{a}_i,$$

$$\partial_+\left(Rv^i\right) + \nabla_j t^{ji} = R\left(\mathfrak{a}^i + \Omega^{ij}v_j\right), \qquad \partial_+(S + \hat s_{diss}) + \nabla_i j^i_s \geq 0, \tag{15}$$

where,

$$\mathfrak{a}_i = e^{\Phi}\left(\partial_i\mathcal{B}_+ - \partial_+\mathcal{B}_i\right), \qquad \Omega_{ij} = \partial_i\mathcal{B}_j - \partial_j\mathcal{B}_i, \tag{16}$$

are frame acceleration and vorticity. They are responsible for pseudo force and power. The currents are given as,

$$\begin{aligned}
&\text{Energy Current:} && j^i_\epsilon = \left(E + P + \frac{1}{2}Rv^i v_i\right)v^i + \varsigma^i_\epsilon + \pi^{ik}v_k,\\
&\text{Stress-Energy Tensor:} && t^{ij} = Pg^{ij} + Rv^i v^j + \pi^{ij},\\
&\text{Entropy Current:} && j^i_s = Sv^i + \varsigma^i_s,
\end{aligned} \tag{17}$$

with dissipation:

Energy Current Dissipation: $\varsigma^i_\epsilon = \mathcal{E}^i$, Momentum Current Dissipation: $\pi^{ij} = \Pi^{ij}$,

Entropy Dissipation: $\hat s_{diss} = \Upsilon_{s-}$, Entropy Current Dissipation: $\varsigma^i_s = \Upsilon^i_s$. (18)

For a leading order fluid on said background we have,

$$\pi^{ij} = -\eta\left(\nabla^i v^j + \nabla^j v^i - \frac{2}{3}g^{ij}\nabla_k v^k\right) - \zeta g^{ij}\nabla_k v^k, \qquad \varsigma^i_\epsilon = \kappa\nabla^i\vartheta + \tilde\omega\epsilon^{ijk}\partial_j v_k,$$

$$\hat s_{diss} = 0, \qquad \varsigma^i_s = \frac{1}{\vartheta}\left[\kappa\nabla^i\vartheta + (\tilde\omega + \vartheta^2 C_1)\epsilon^{ijk}\partial_j v_k\right]. \tag{19}$$

We hence see that thermodynamic variables of null fluid can be realized as the thermodynamic variables of non-relativistic fluid.

Conclusions

In this poster we have explained the mechanism to get non-relativistic fluid through light cone reduction of null fluid. One of the most striking features of this construction is that the null fluid is entirely equivalent to non-relativistic fluid, and is related just by compactification of the null isometry direction. Thus all the features of non-relativistic hydrodynamics – symmetries, constitutive relations, thermodynamics, entropy current, equilibrium partition function etc. are in one to one correspondence with the null fluid. This gives us a new and rather simplified way to look at non-relativistic fluids altogether, since we have all the machinery of relativistic hydrodynamics at our disposal. The null fluid construction can also be extended to study most generic charged anomalous non-relativistic fluid on arbitrary backgrounds.

Please look at our paper [arXiv:1505.05677] for references.