HIGHER-FORM SYMMETRIES AND TOPOLOGICAL PHASE TRANSITIONS

Akash Jain

University of Amsterdam

[2301.09628] Armas, AJ

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MOTIVATION

- **Symmetries** are powerful guiding principle for developing effective theories for physical systems without a detailed understanding of their microscopic constituents.
- Equilibrium phases of matter can be organised according to their symmetries and whether these are **spontaneously broken** or **unbroken** in the ground state, commonly known as the **Landau paradigm**.
- Symmetries can even be useful when they are only **approximately** respected by the system.
  The canonical example comes from *chiral perturbation theory*, where pions are seen as pseudo-Goldstones of approximate SU(2) chiral symmetry.
  Other examples: *pinned crystals, pinned charge density waves, pseudo-superfluids* etc.

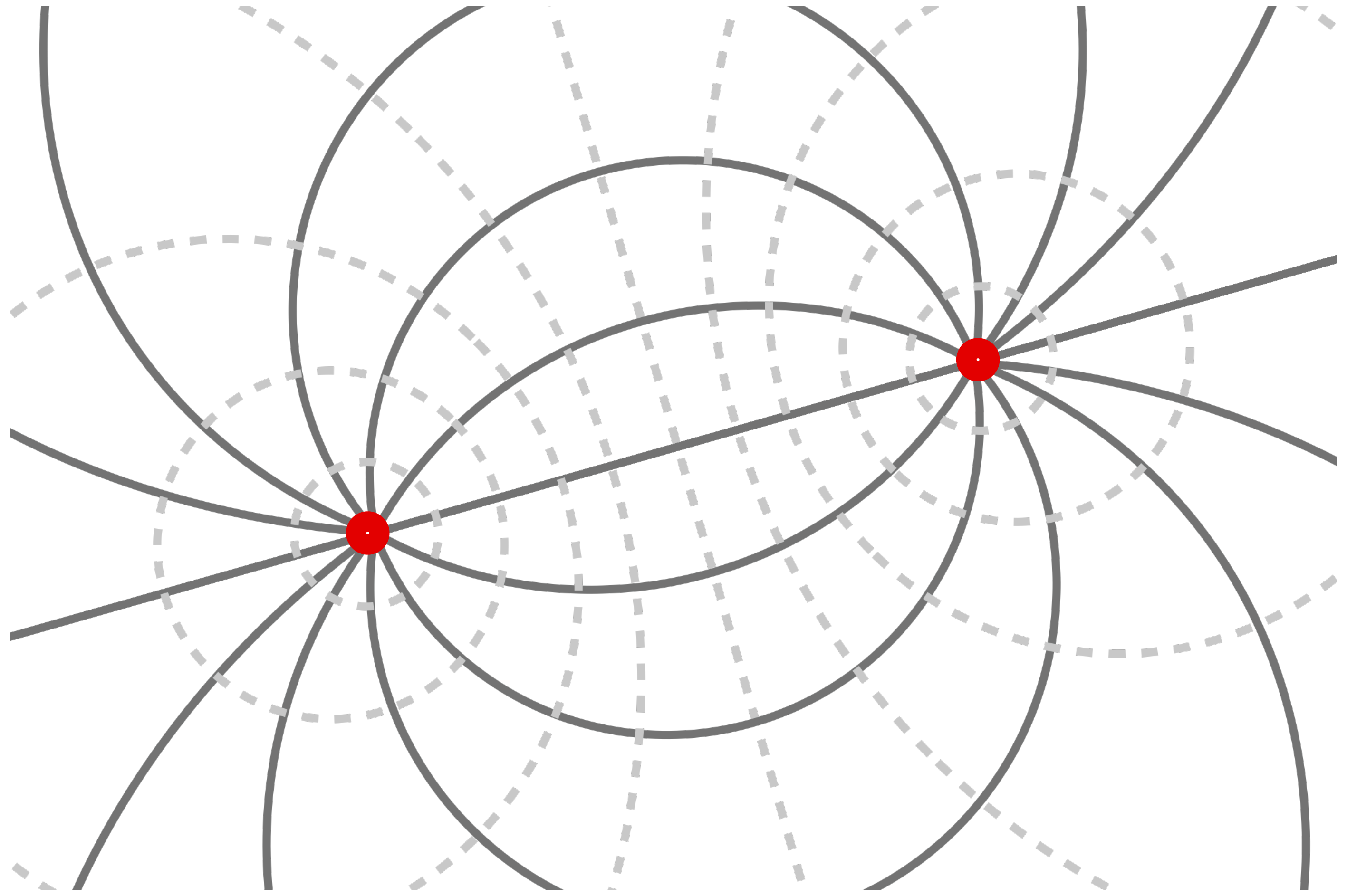
MOTIVATION

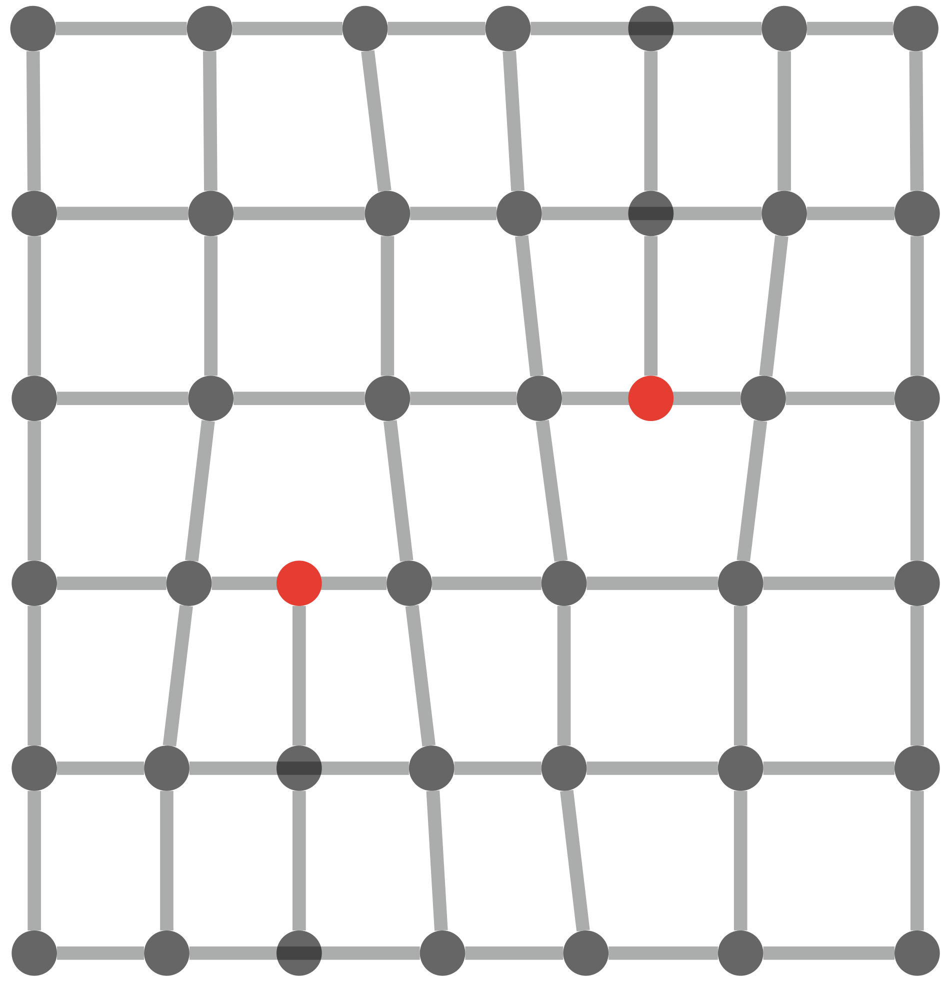
- In recent years, the notion of symmetries has been **generalised** to include higher-form symmetries, higher-group symmetries, subsystem symmetries, and non-invertible symmetries.
- These allow for a **generalised Landau paradigm**, that also include exotic phases of matter, such as *topologically ordered states, spin liquids, fractons, topological insulators,* etc.
- The focus of this talk is **continuous higher-form symmetries**, which concerns higher-dimensional charged objects, such as strings and surfaces.
- These describe **topological order** in many-body systems, such as *equipotential planes* in a superfluid, *lattice planes* in a crystal, *magnetic fields* in a plasma, or *electric fields* in a dielectric fluid.

Gaiotto, Kapustin, Seiberg, Willett [1412.5148]

MOTIVATION

- Explicit breaking of higher-form symmetries describes **topological defects**, such as *superfluid vortices*, *crystal dislocations*, *magnetic monopoles*, or *free charges*.
- Topological defects mediate **topological phase transitions**,[1] wherein a spontaneously broken symmetry gets restored. Examples include *superfluid phase transition*, *melting*, and *plasma phase transition*.





[1]Not to be confused with phase transitions between topologically ordered phases.

HIGHER-FORM SYMMETRIES

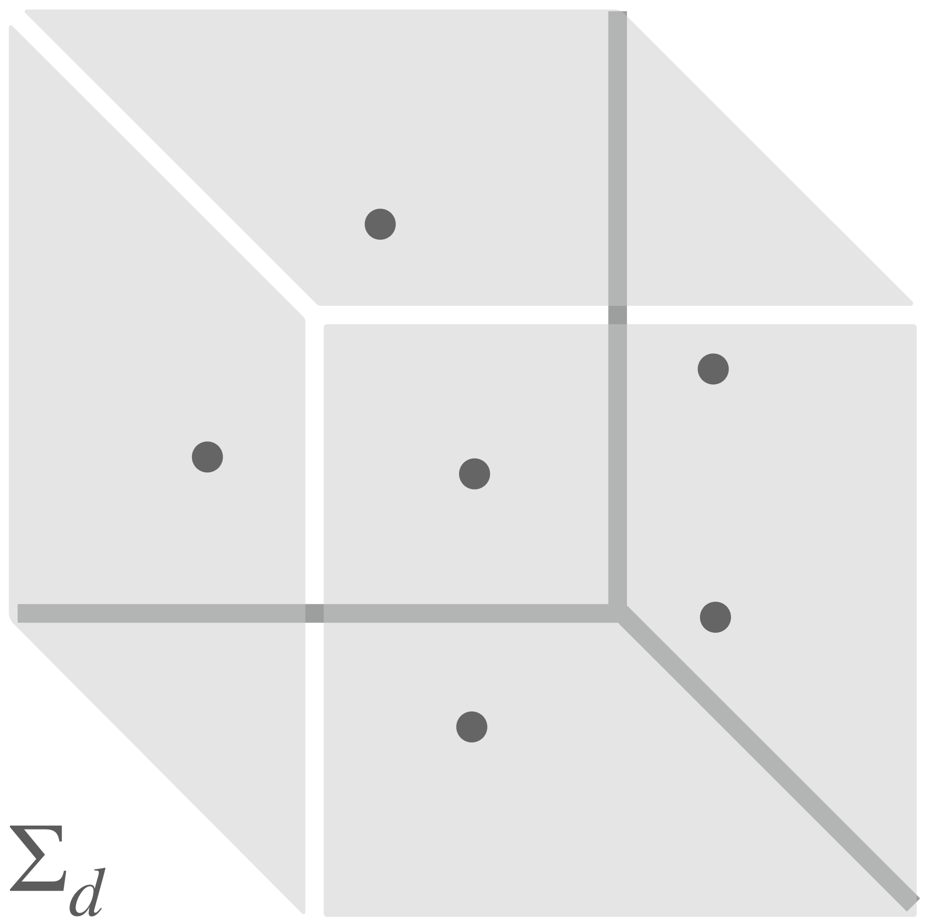
and their breaking

HIGHER-FORM SYMMETRIES

- Continuous 0-form symmetry:

$$\partial_\mu J^\mu = 0 \qquad \Longrightarrow \qquad \partial_t J^t + \partial_i J^i = 0$$

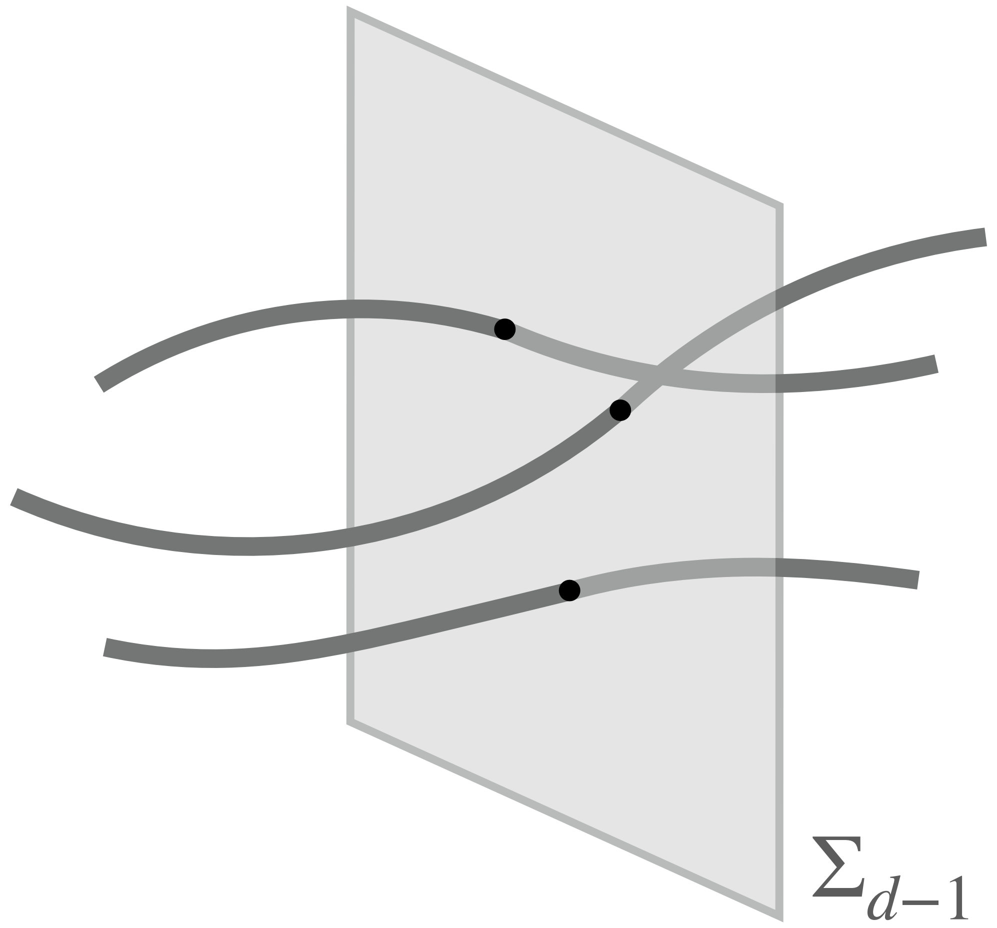
- The number of charged particles in a volume Σ_d is conserved in **time**.



- Continuous 1-form symmetry:

$$\partial_\mu J^{\mu\nu} = 0 \qquad \Longrightarrow \qquad \begin{aligned} \partial_t J^{ti} + \partial_k J^{ki} &= 0 \\ \partial_i J^{ti} &= 0 \end{aligned}$$

- The number of charged "strings" passing a cross section Σ_{d-1} are conserved in **time** and under **spatial deformations** of Σ_{d-1}.

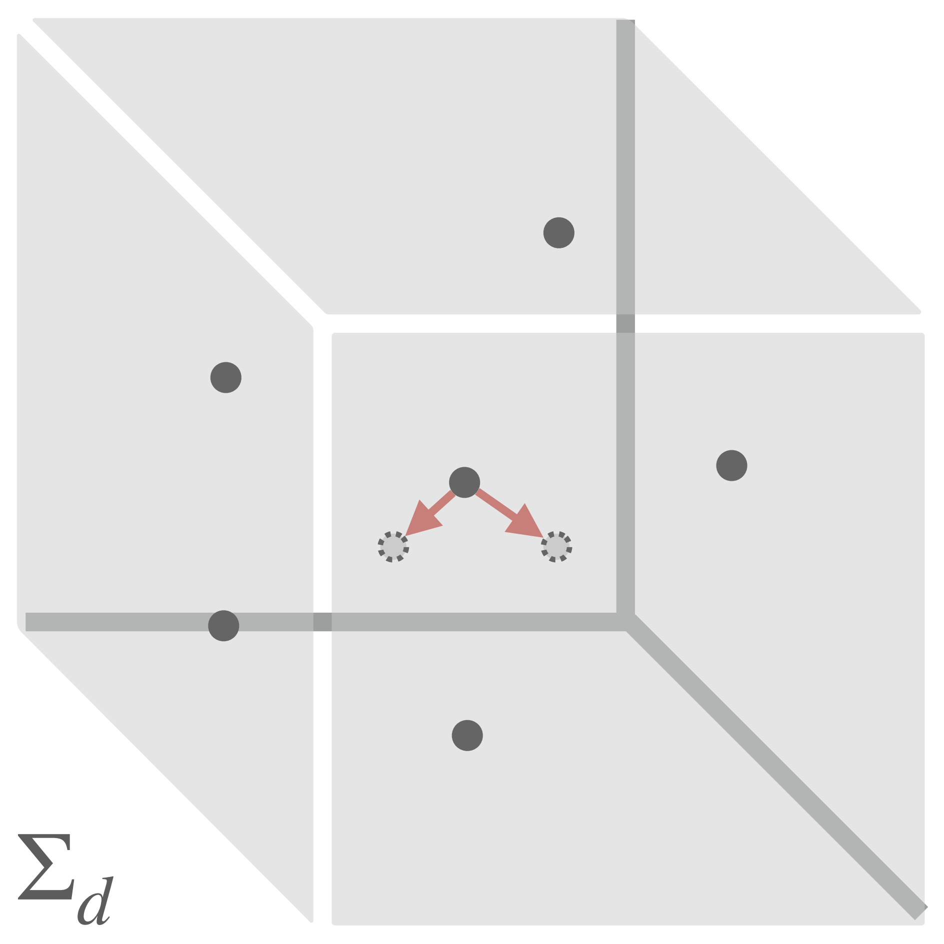


APPROXIMATE HIGHER-FORM SYMMETRIES

- Continuous approximate 0-form symmetry:

$$\partial_\mu J^\mu = -\ell L$$

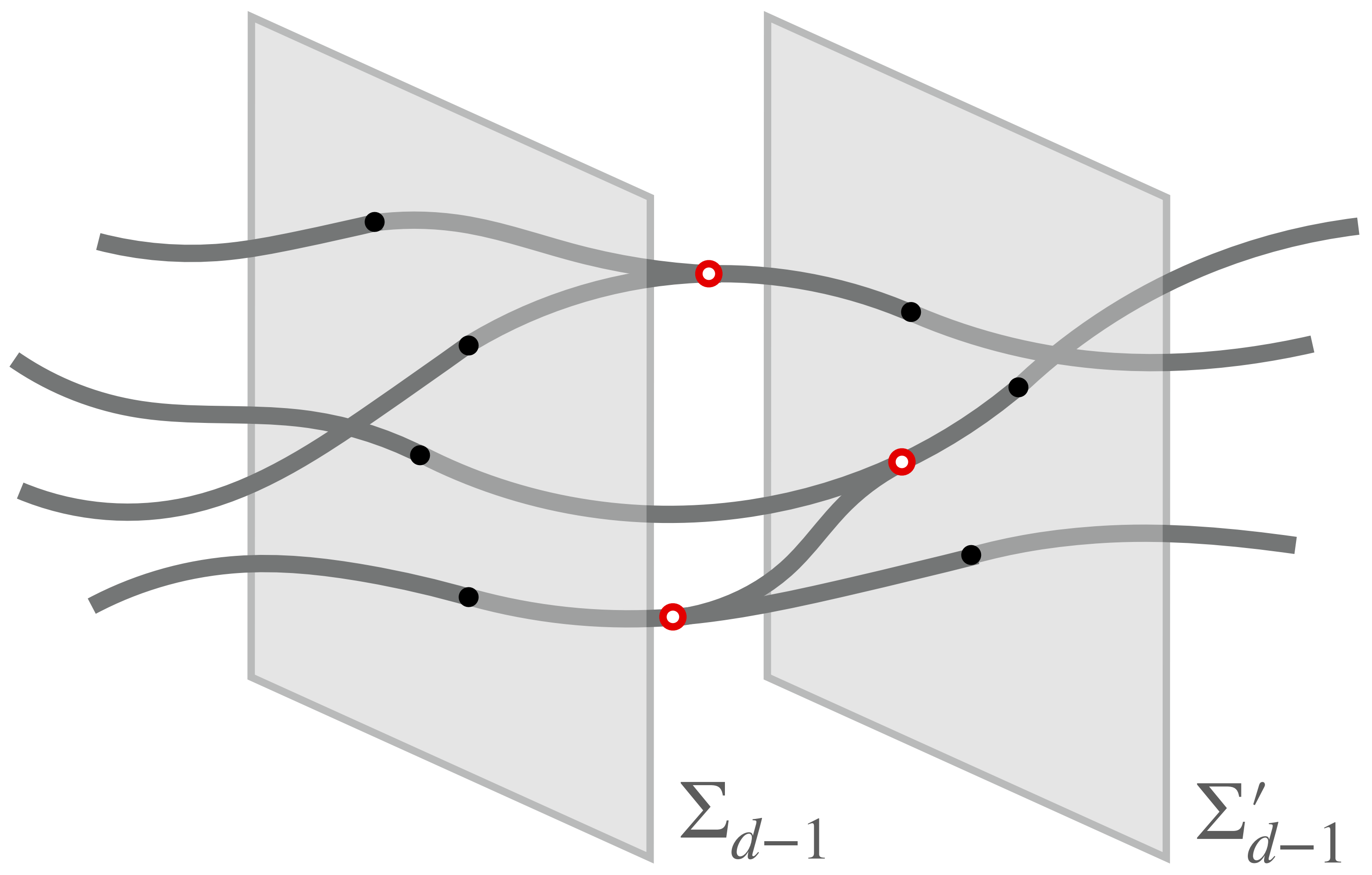
- Charged particles in a volume Σ_d can be created/annihilated in **time**.



- Continuous approximate 1-form symmetry:

$$\partial_\mu J^{\mu\nu} = -\ell L^\nu$$

- Charged “strings” passing a cross section Σ_{d-1} can be created/annihilated in **time** and under **spatial deformations** of Σ_{d-1}.

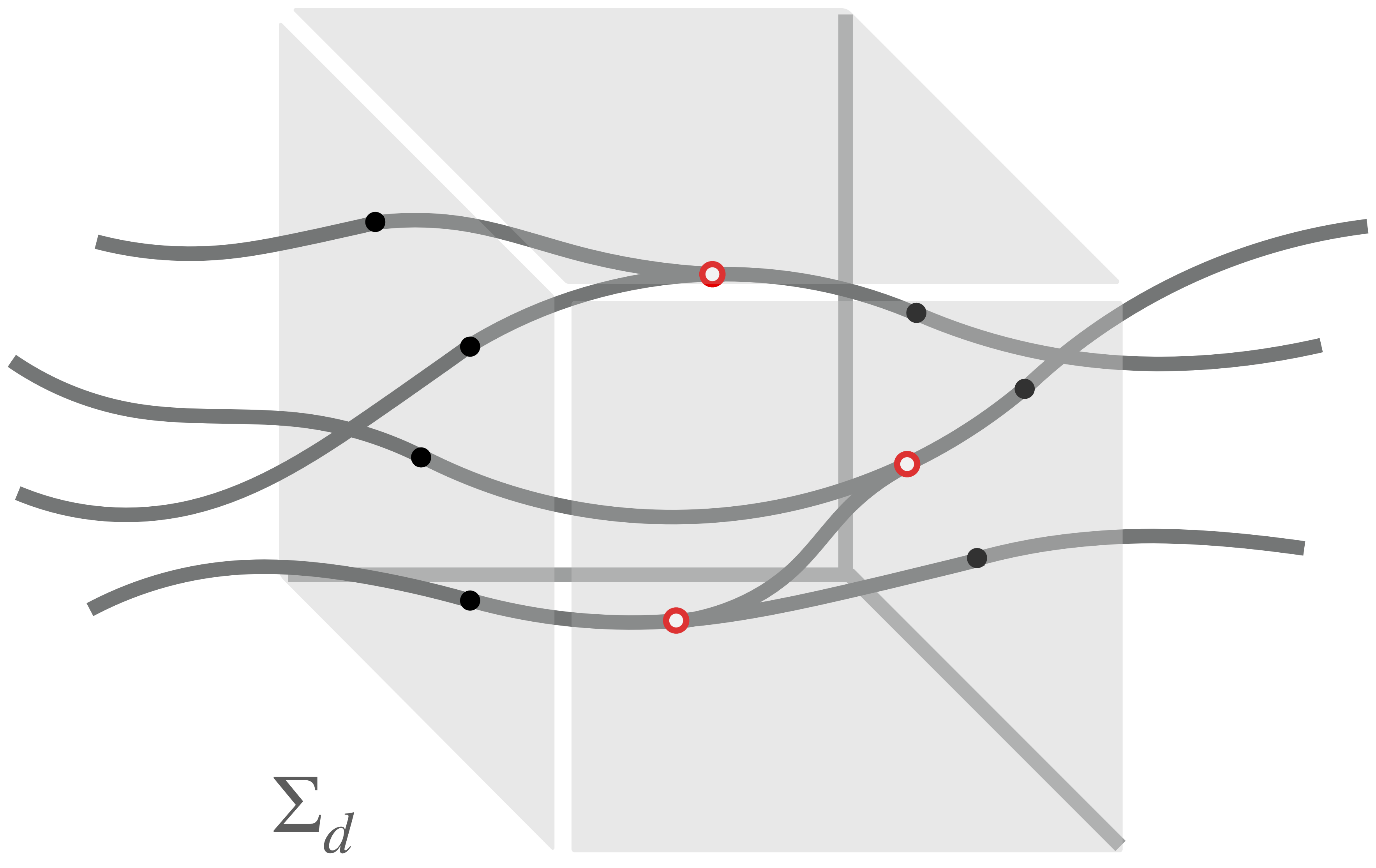


APPROXIMATE HIGHER-FORM SYMMETRIES

- The defects of a 1-form symmetry themselves furnish an emergent 0-form “defect” symmetry

$$\partial_\mu L^\mu = 0$$

- The number of defects in a volume Σ_d is conserved in **time**.



BACKGROUND FIELDS

- We can introduce a 1-form gauge field A_μ and a background phase Φ to probe an approximate 0-form symmetry

$$\delta S[A,\Phi] = \int \mathrm{d}^d x \left(J^\mu \, \delta A_\mu + \ell L \, \delta\Phi \right)$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

$$\Phi \to \Phi - \Lambda$$

- Similarly, we can introduce a 2-form gauge field $A_{\mu\nu}$ and a background phase Φ_μ to probe an approximate 1-form symmetry

$$\delta S[A,\Phi] = \int \mathrm{d}^d x \left(\frac{1}{2} J^{\mu\nu} \, \delta A_{\mu\nu} + \ell L^\mu \, \delta\Phi_\mu \right)$$

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

$$\Phi_\mu \to \Phi_\mu - \Lambda_\mu + \partial_\mu \Lambda_\ell$$

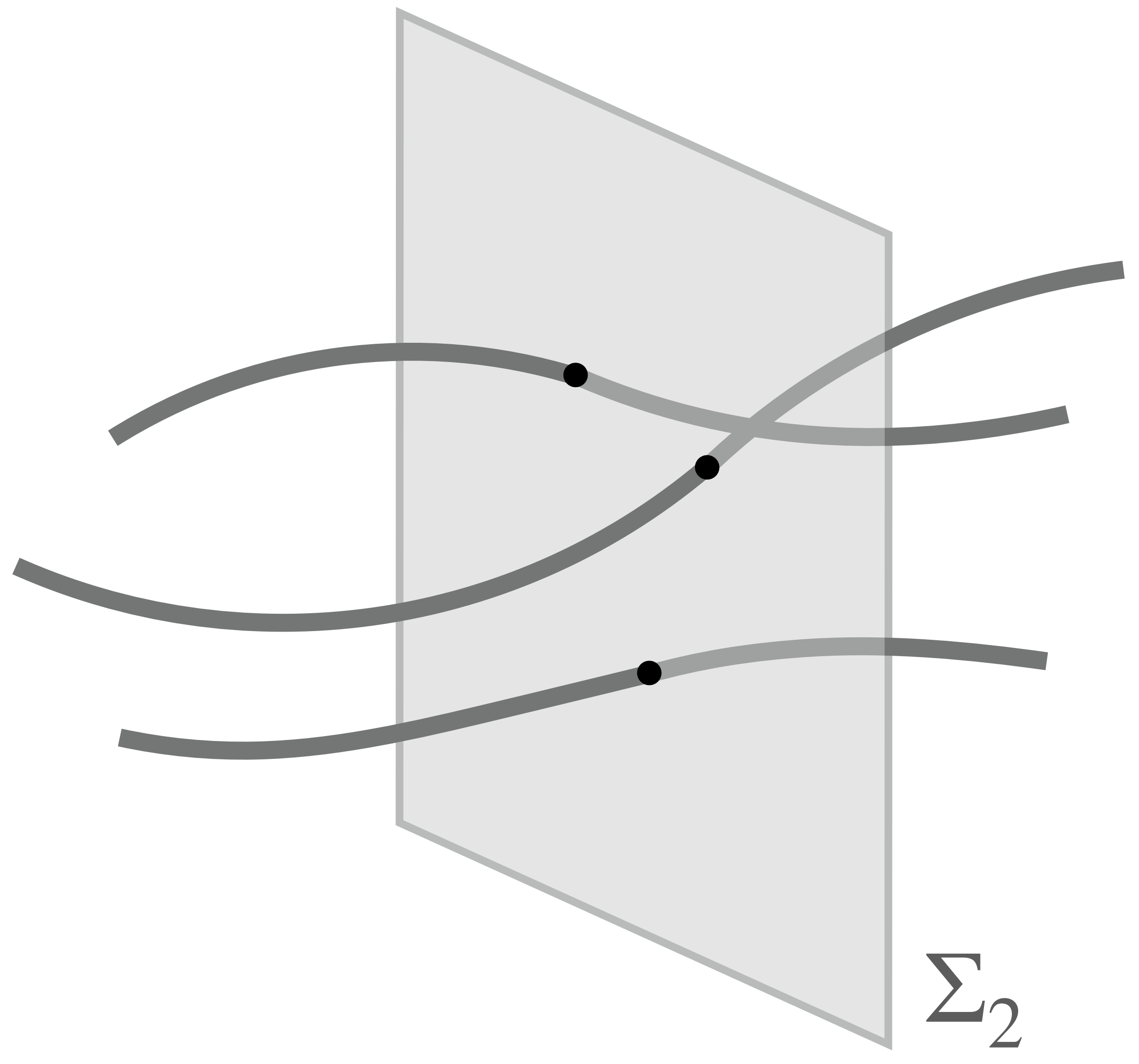
$$\Lambda_\mu \to -\partial_\mu \Lambda_\ell$$

EXAMPLE: ELECTROMAGNETISM

- $d = 3$ electromagnetism in vacuum

$$S = -\int \mathrm{d}^4x \left(\frac{1}{4} \mathscr{F}^{\mu\nu} \mathscr{F}_{\mu\nu} \right) \qquad \Longrightarrow \qquad \partial_\mu \mathscr{F}^{\mu\nu} = 0$$

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu \qquad \qquad \partial_\mu \star \mathscr{F}^{\mu\nu} = 0$$



- It has two 1-form symmetries:

$$J^{\mu\nu} = -\mathscr{F}^{\mu\nu} \qquad \qquad \tilde{J}^{\mu\nu} = \star \mathscr{F}^{\mu\nu}$$

The associated charged objects are **electric** and **magnetic field lines**.

- These symmetries also persist in the presence of polarised/dielectric matter

$$J^{\mu\nu} = -\mathscr{F}^{\mu\nu} + \mathscr{M}^{\mu\nu}$$

EXAMPLE: ELECTROMAGNETISM

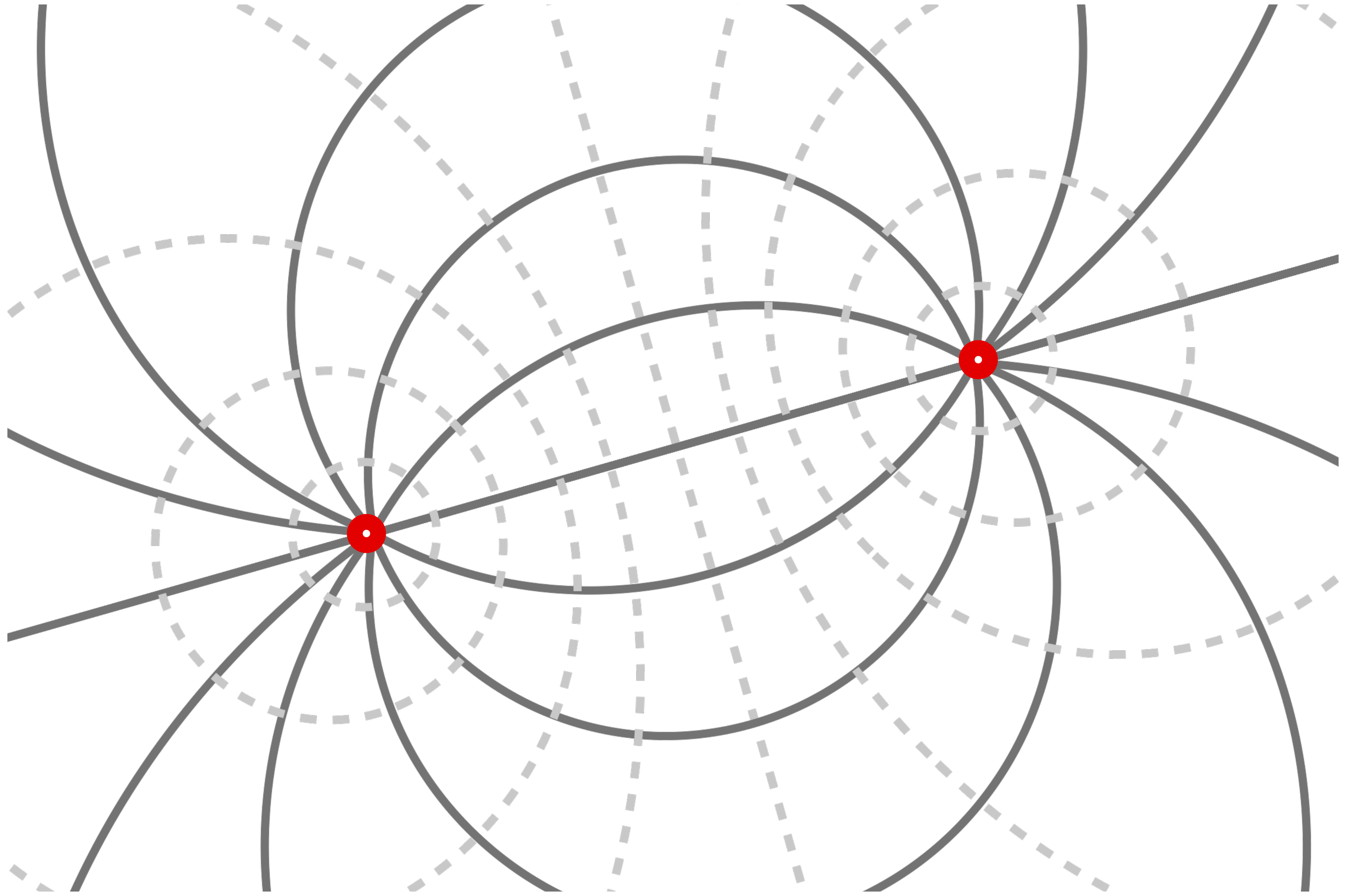
- In the presence of **free electric charges**, the electric 1-form symmetry is **explicitly broken**, but the magnetic 1-form symmetry persists.

$$S = -\int d^4x \left(\frac{1}{4}\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} + D_\mu\Psi^* D^\mu\Psi + V(\Psi^*\Psi) \right) \qquad D_\mu = \partial_\mu - i\ell\mathscr{A}_\mu$$

$$J^{\mu\nu} = -\mathscr{F}^{\mu\nu}$$

$$\tilde{J}^{\mu\nu} = \star\mathscr{F}^{\mu\nu} \qquad \Longrightarrow \qquad \partial_\mu J^{\mu\nu} = -\ell L^\nu, \quad \partial_\mu \tilde{J}^{\mu\nu} = 0$$

$$L^\mu = i\left(\Psi^* D^\mu\Psi - D^\mu\Psi^*\Psi\right)$$



- Similarly, breaking of the magnetic 1-form symmetry amounts to the introduction of **magnetic monopoles**.

EXAMPLE: ELECTROMAGNETISM

- We can manifest the electric 1-form symmetry via

$$S[A, \Phi] = -\int \mathrm{d}^4x \left(\frac{1}{4}\xi^{\mu\nu}\xi_{\mu\nu} + \mathrm{D}_\mu\Psi^* \mathrm{D}^\mu\Psi + V(\Psi^*\Psi) \right)$$

$$\xi_{\mu\nu} = \mathscr{F}_{\mu\nu} + A_{\mu\nu}$$

$$\psi_\mu = \ell\left(\mathscr{A}_\mu - \Phi_\mu\right)$$

$$\mathrm{D}_\mu = \partial_\mu - i\psi_\mu$$

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu \qquad \mathscr{A}_\mu \to \mathscr{A}_\mu - \Lambda_\mu$$

$$\Phi_\mu \to \Phi_\mu - \Lambda_\mu + \partial_\mu\Lambda_\ell \qquad \Psi \to \mathrm{e}^{-i\ell\Lambda_\ell}\,\Psi$$

Electromagnetism is **1-form superfluidity**: 1-form symmetry is **spontaneously** broken. Free electric charges arise as **explicit breaking** of the 1-form symmetry.

- The "local" U(1) symmetry of electromagnetism: $\Lambda_\mu = -\partial_\mu\lambda,\ \Lambda_\ell = -\lambda$.
- It is also possible to manifest the magnetic 1-form symmetry.[1]

[1]Both higher-form symmetries cannot be gauged together on account of a mixed 't Hooft anomaly.

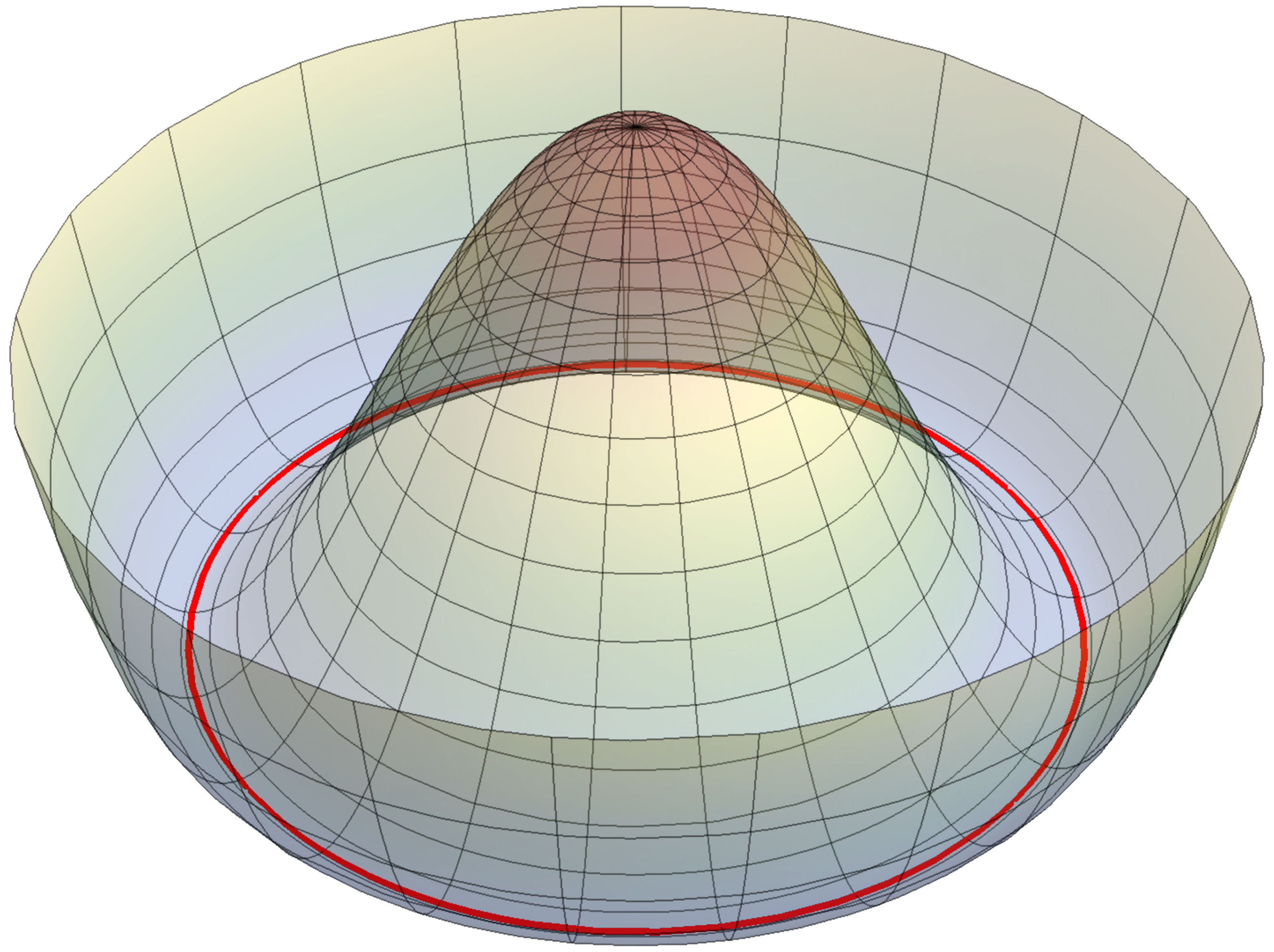
EXAMPLE: SUPERCONDUCTIVITY

- In the **superconducting** phase, Ψ condenses around a nonzero vev $|\Psi| = m/\sqrt{2}$.
- The phase fluctuations around the vev are described by the **Ginzburg-Landau theory**

$$S[A,\Phi] = -\int d^4x \left(\frac{1}{4}\xi^{\mu\nu}\xi_{\mu\nu} + \frac{1}{2}m^2\psi_\mu\psi^\mu + \ldots \right)$$

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu \qquad \mathscr{A}_\mu \to \mathscr{A}_\mu - \Lambda_\mu$$

$$\Phi_\mu \to \Phi_\mu - \Lambda_\mu + \partial_\mu\Lambda_\ell \qquad \phi_\ell \to \phi_\ell - \Lambda_\ell$$



- In the superconducting (Higgs) phase, the 1-form symmetry is **spontaneously** and weakly **explicitly** broken.
The emergent 0-form “defect” symmetry is also **spontaneously** broken.

$$\Psi = \frac{m}{\sqrt{2}} e^{i\ell\phi_\ell}$$

$$\psi_\mu = \ell\left(\mathscr{A}_\mu - \Phi_\mu - \partial_\mu\phi_\ell\right)$$

HIGHER-FORM SYMMETRIES

- In general spatial dimensions d, electromagnetism has an **electric 1-form** and **magnetic $(d-2)$-form** symmetry. The defects are free electric charges and magnetic monopoles.
- Electromagnetism can be viewed as a 1-form or $(d-2)$-form superfluid.[1]
- Ordinary 0-form superfluids have a $(d-1)$-form symmetry, with the defects being **vortices**.[2]

$$J^{\mu\nu\cdots} = \epsilon^{\lambda\mu\nu\cdots}\partial_\lambda \phi$$

- Crystals also have a $(d-1)$-form symmetry, with the defects being **dislocations**.[3]

$$J^{I\mu\nu\cdots} = \epsilon^{\lambda\mu\nu\cdots}\partial_\lambda \phi^I$$

[1] Hofman, Iqbal [1802.09512]; Armas, AJ [1808.01939, 1811.04913]
[2] Delacrétaz, Hofman, Mathys [1908.06977]
[3] Grozdanov, Poovuttikul [1801.03199]; Armas, AJ [1908.01175]; Armas, Heumen, AJ, Lier [2211.02117]

HIGHER-FORM FLUIDS

with approximate higher-form symmetry

© Mildred Thompson "Magnetic Fields"

THERMAL EQUILIBRIUM

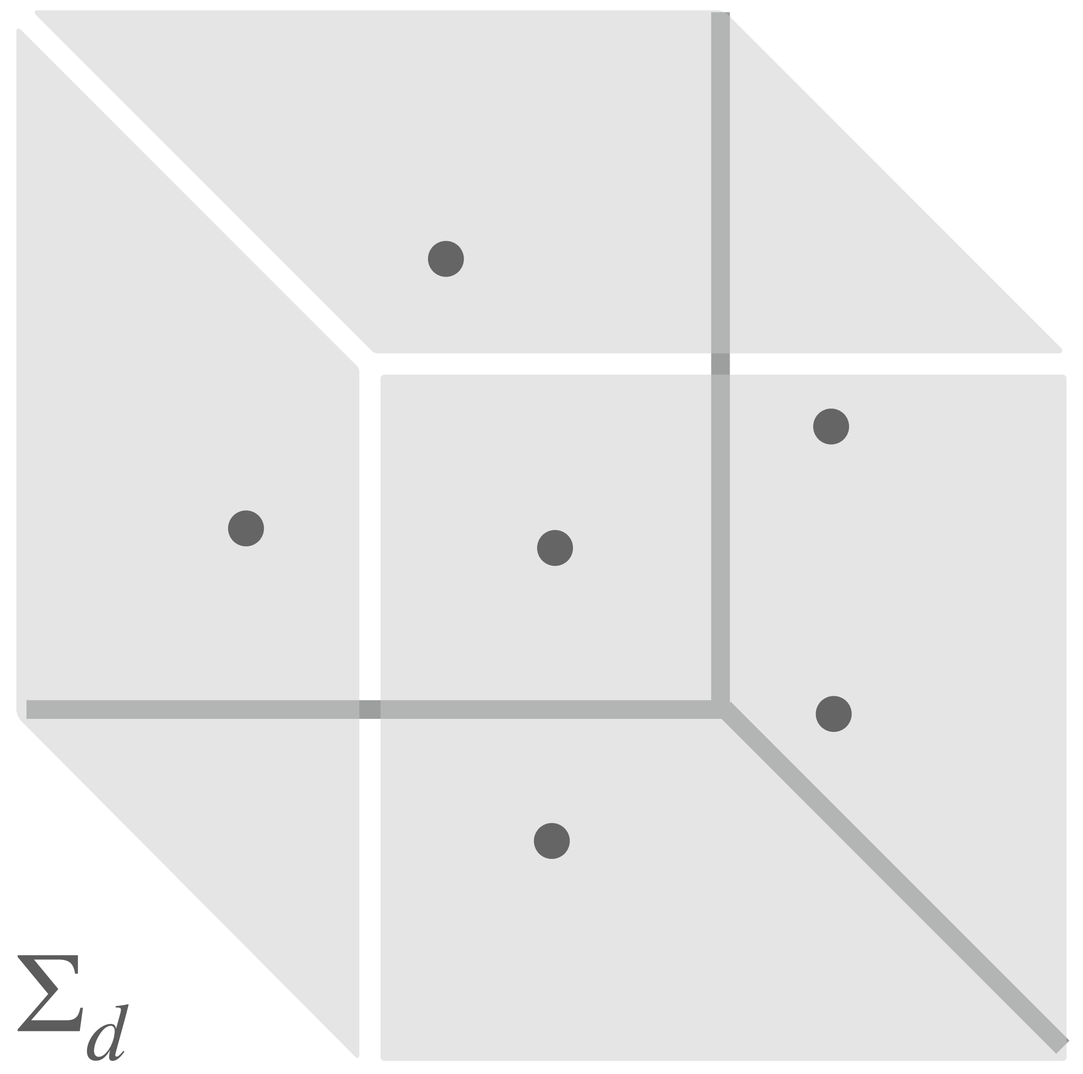
- Many-body systems at thermal equilibrium can be characterised by their **thermal partition function**.
- Thermal partition function is a functional of background fields and can be used to obtain equilibrium values of (approximately) conserved densities and fluxes.
- For systems with **spontaneously unbroken** symmetries, the thermal partition function is a "local" functional of background fields.
- For systems with **spontaneously broken** symmetries, the thermal partition function is "non-local", and is given by a functional integral over the time-independent configurations of the **Goldstone fields**.

THERMAL EQUILIBRIUM: 0-FORM HYDROSTATICS

- For a 0-form symmetry, a thermal ensemble with constant charge density is described by a **thermal partition function**

$$\mathcal{Z}[A] = \exp \int \mathrm{d}^d x \left(\frac{1}{2} \chi \mu^2 + \ldots \right) \qquad \mu = \mu_0 + A_t$$

$$\implies \qquad J^t = n = \chi \mu \qquad J^i = 0$$



- This works because A_t is invariant under **time-independent gauge transformations**

$$A_t \to A_t + \partial_t \Lambda$$

This is no longer true for higher-form symmetries

$$A_{ti} \to A_{ti} + \partial_t \Lambda_i - \partial_i \Lambda_t$$

THERMAL EQUILIBRIUM: 1-FORM HYDROSTATICS

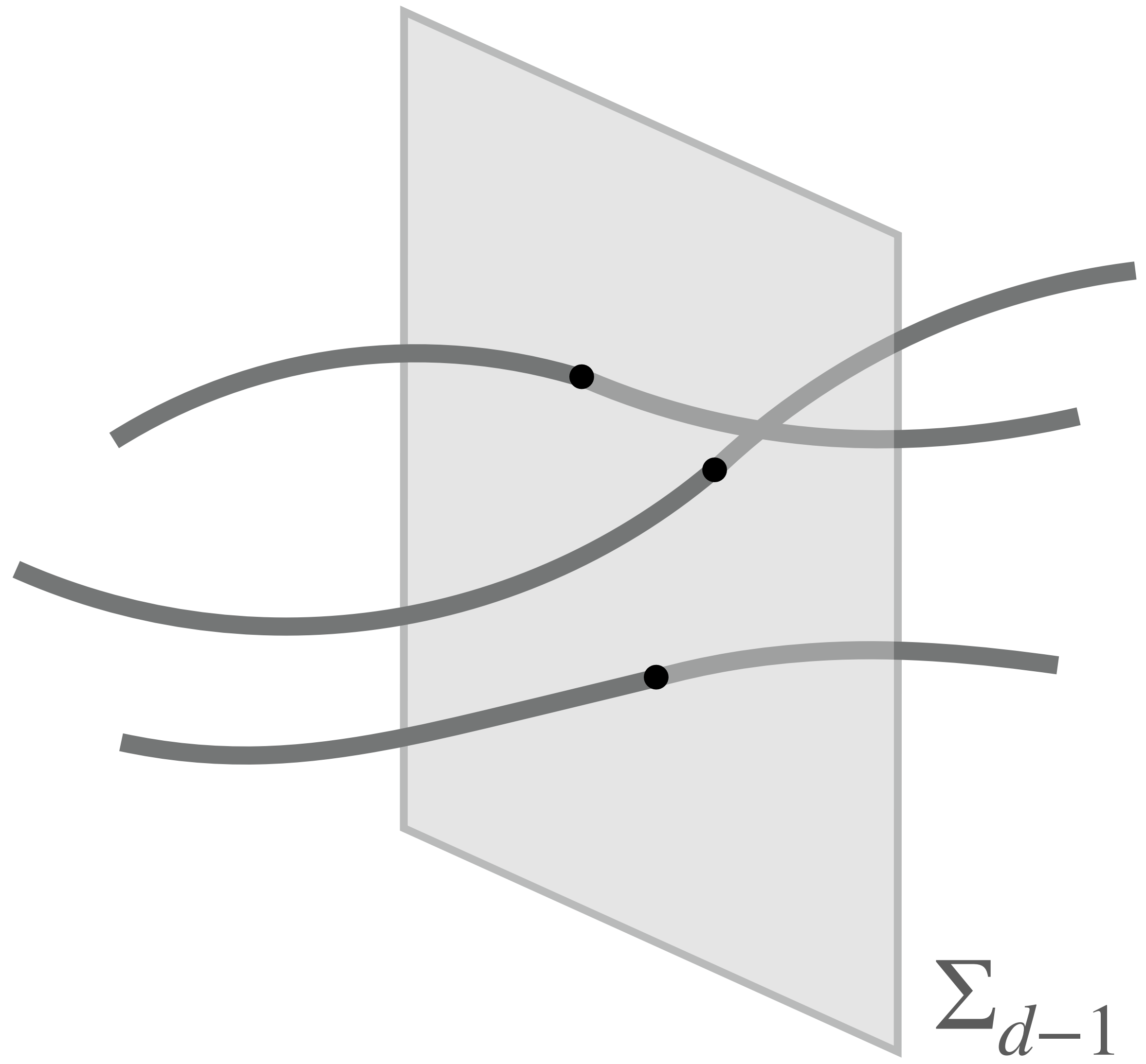
- We need to **partially-spontaneously break** the higher-form symmetry in the time-direction

$$\varphi \to \varphi - \Lambda_t \qquad \mu_i = -\partial_i \varphi + A_{ti}$$

This allows us to construct a "non-local" partition function

$$\mathscr{Z}[A] = \int \mathscr{D}\varphi \exp \int \mathrm{d}^d x \left(\frac{1}{2} \chi \mu_i \mu^i + \ldots \right)$$

$$\implies \qquad J^{ti} = n^i = \chi \mu^i \qquad J^{ij} = 0$$



- Classical configuration equation of φ implements the **Gauss constraint**

$$\partial_i J^{ti} = 0 \implies \partial_i \partial^i \varphi = 0$$

EXPLICITLY-BROKEN SYMMETRIES

- The partition function can also depend on Φ_μ through the **defect chemical potential**

$$\mu_\ell = -\ell\left(\varphi - \Phi_t\right)$$

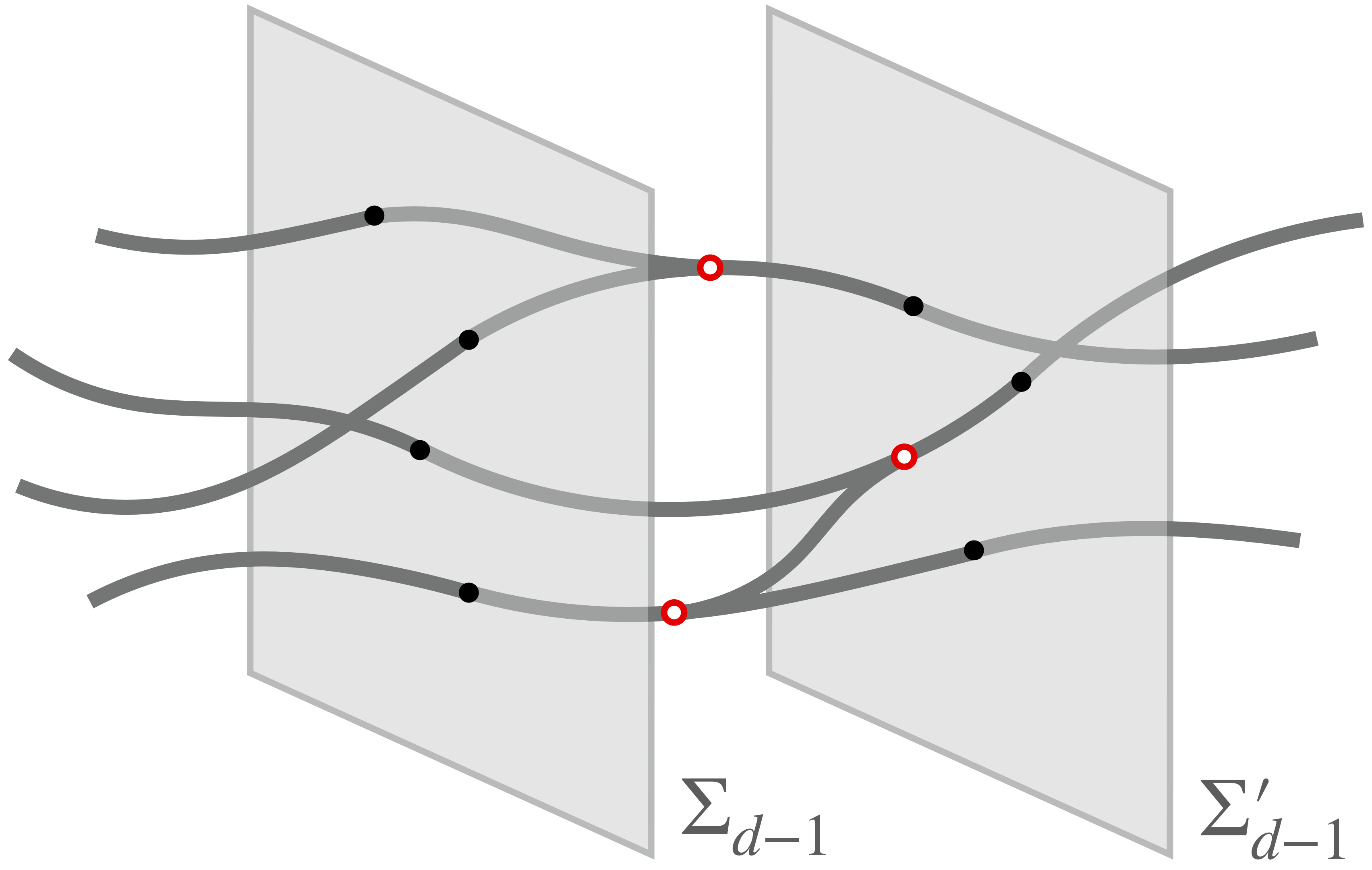
$$\Phi_t \to \Phi_t + \partial_t \Lambda_\ell$$

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int \mathrm{d}^d x \left(\frac{1}{2}\chi\,\mu_i\mu^i + \frac{1}{2}\chi_\ell\,\mu_\ell^2 + \ldots \right)$$

$$\Longrightarrow \qquad J^{ti} = n^i = \chi\,\mu^i \qquad J^{ij} = 0$$

$$L^t = n_\ell = \chi_\ell\,\mu_\ell \qquad L^i = 0$$



- Classical configuration equation for φ imposes the **Gauss constraint**

$$\partial_i J^{ti} = \ell L^t \Longrightarrow \partial_i\partial^i\varphi = k_0^2\varphi$$

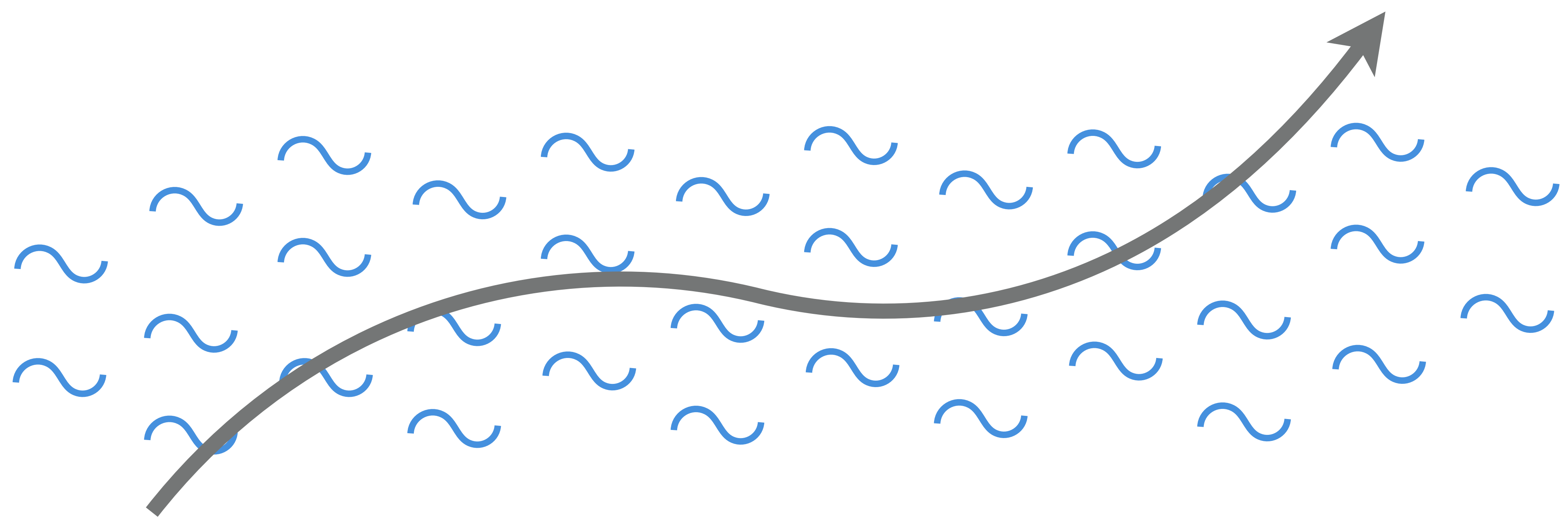
$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2\chi_\ell}}$$

"Debye length"

Armas, AJ [2301.09628]

HYDRODYNAMICS

- **Hydrodynamics** is a framework to capture perturbative departures of a many-body system from thermal equilibrium.
- The relevant hydrodynamic degrees of freedom are a set of **symmetry parameters** corresponding to each global symmetry (conserved charge) of the system.
- Additionally, we need to add massless **Goldstone fields** for each spontaneously broken global symmetry.



0-FORM HYDRODYNAMICS

- The hydrodynamic description is based on the **conservation laws**

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$$

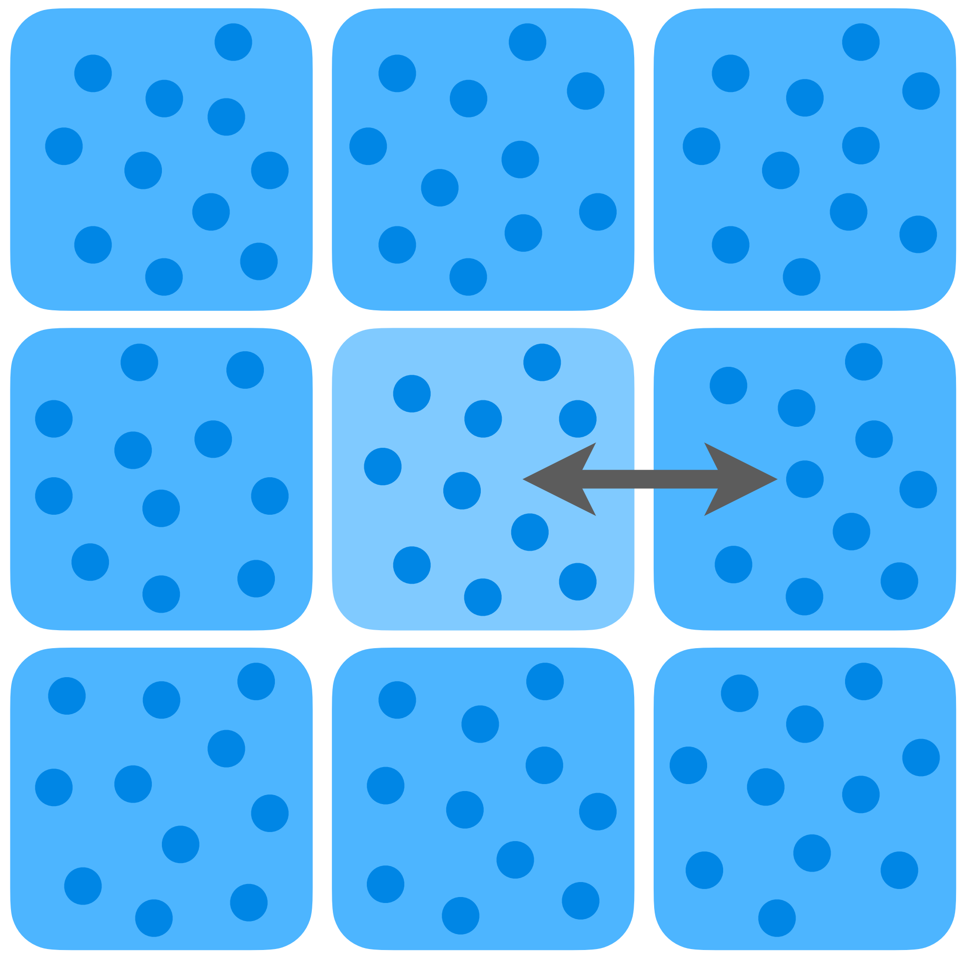
$$\nabla_\mu J^\mu = -\ell L, \qquad \nabla_\mu T^{\mu\nu} = F^{\nu\rho}J_\rho + \Xi^\nu L$$

$$\Xi_\mu = \partial_\mu \Phi + A_\mu$$

  we have allowed for violation of the 0-form symmetry.

- The **hydrodynamic fields** are Λ_β, β^μ transforming as

$$\delta\Lambda_\beta = \pounds_\chi \Lambda_\beta - \pounds_\beta \Lambda, \qquad \delta\beta^\mu = \pounds_\chi \beta^\mu$$



- These can be used to define gauge-invariant hydrodynamic fields μ, T, u^μ as

$$\frac{\mu}{T} = \Lambda_\beta + \beta^\mu A_\mu, \qquad \frac{u^\mu}{T} = \beta^\mu$$

Armas, AJ, Lier [2112.14373]

0-FORM HYDRODYNAMICS

- Hydrodynamics is characterised by its **constitutive relations**

$$J^\mu, \quad L, \quad T^{\mu\nu} \qquad \text{in terms of} \qquad \mu, \quad T, \quad u^\mu, \qquad A_\mu, \quad \Phi, \quad g_{\mu\nu}$$

- Imposing the **second law of thermodynamics**, we find the constitutive relations

$$J^\mu = n\, u^\mu - \sigma\, P^{\mu\nu} \left(T\partial_\nu \frac{\mu}{T} + u^\lambda F_{\lambda\nu} \right)$$

$$L = -\,\ell \sigma_\ell \left(u^\mu \Xi_\mu - \mu \right)$$

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu} - 2\eta\, P^{\mu\rho} P^{\nu\sigma} \nabla_{\langle\rho} u_{\sigma\rangle} - \zeta\, P^{\mu\nu} \nabla_\lambda u^\lambda$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$\delta p = s\delta T + n\delta\mu$$

$$\epsilon = Ts + \mu n - p$$

$$\sigma, \sigma_\ell, \eta, \zeta \geq 0$$

- The coefficient σ_ℓ results in **charge relaxation**

$$\nabla_\mu J^\mu = -\ell L \qquad \Longrightarrow \qquad u^\mu \partial_\mu n + \ldots = -\ell^2 \sigma_\ell\, \mu(n) + \ldots$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the **fluid sound** and **shear modes**

$$u_\parallel, T: \quad \omega = \pm v_s k - \frac{i}{2} D_\pi^\parallel k^2 + \dots$$

$$u_\perp: \quad \omega = -iD_\pi^\perp k^2 + \dots$$

$$v_s^2 = \frac{\partial p}{\partial \epsilon}$$

$$D_\pi^\parallel = \frac{\zeta + 2\frac{d-1}{d}\eta}{\epsilon + p}, \qquad D_\pi^\perp = \frac{\eta}{\epsilon + p}$$

- The charge fluctuations give rise to a **diffusive mode**

$$\mu: \quad \omega = -iD_n k^2 - i\Gamma$$

$$D_n = \frac{\sigma}{\chi}, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}, \qquad \chi = \frac{\partial n}{\partial \mu}$$

Charge fluctuations are **damped** due to explicit symmetry breaking.

1-FORM HYDRODYNAMICS

- The hydrodynamic description is based on the **conservation laws**

$$\nabla_\mu J^{\mu\nu} = \ell L^\nu \qquad \nabla_\mu T^{\mu\nu} = \frac{1}{2} F^{\nu\rho\sigma} J_{\rho\sigma} + \ell \Xi^{\nu\rho} L_\rho \qquad F_{\mu\nu\rho} = 3\partial_{[\mu} A_{\nu\rho]}$$

$$\nabla_\mu L^\mu = 0 \qquad \Xi_{\mu\nu} = 2\partial_{[\mu}\Phi_{\nu]} + A_{\mu\nu}$$

- The **hydrodynamic fields** are Λ^β_μ, Λ^β_ℓ, β^μ, together with φ, transforming as

$$\delta\Lambda^\beta_\mu = \pounds_\chi \Lambda^\beta_\mu - \pounds_\beta \Lambda_\mu \qquad \delta\beta^\mu = \pounds_\chi \beta^\mu \qquad \delta\varphi = \pounds_\chi \varphi - \beta^\mu \Lambda_\mu$$

$$\delta\Lambda^\beta_\ell = \pounds_\chi \Lambda^\beta_\ell - \pounds_\beta \Lambda_\ell$$

- These can be used to define the gauge-invariant hydrodynamic fields

$$\frac{\mu_\mu}{T} = \Lambda^\beta_\mu + \beta^\lambda A_{\lambda\mu} - \partial_\mu \varphi, \qquad \frac{\mu_\ell}{T} = -\ell\left(\varphi - \beta^\mu \Phi_\mu - \Lambda^\beta_\ell\right), \qquad \frac{u^\mu}{T} = \beta^\mu$$

Grozdanov, Hofman, Iqbal [1610.07392]; Armas, AJ [1808.01939, 1811.04913, 2301.09628]

JOSEPHSON EQUATION FOR TEMPORAL GOLDSTONE

- The dynamics of φ is governed by a **Josephson equation** of the form

$$\pounds_\beta \varphi = \beta^\mu \Lambda^\beta_\mu + \ldots$$

- We can absorb possible corrections to this equation by redefining Λ^β_μ. This implies

$$u^\mu \mu_\mu = u^\mu \Lambda^\beta_\mu - u^\mu \partial_\mu \varphi = 0$$

Therefore, the 1-form chemical potential is **purely spatial**.

GAUGE REDUNDANCY

- There is a **gauge redundancy** in the description that can be obtained by setting

$$\Lambda_\mu = -\partial_\mu \lambda, \qquad \Lambda_\ell = -\lambda$$

which leaves the background fields $A_{\mu\nu}$, Φ_μ invariant.

- The dynamical fields transform as

$$\delta_\lambda \Lambda^\beta_\mu = \partial_\mu \pounds_\beta \lambda \qquad \delta_\lambda \beta^\mu = 0 \qquad \delta_\lambda \varphi = \pounds_\beta \lambda$$

$$\delta_\lambda \Lambda^\beta_\ell = \pounds_\beta \lambda$$

The physical hydrodynamic fields μ_μ, μ_ℓ, T, u^μ are invariant under these gauge transformations.

1-FORM HYDRODYNAMICS

- The **constitutive relations** are a straight-forward generalisation of the 0-form case

$$J^{\mu\nu} = 2u^{[\mu}n^{\nu]} - \sigma P^{\mu\rho}P^{\nu\sigma}\left(2T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} + u^{\lambda}F_{\lambda\rho\sigma}\right)$$

$$P^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu}$$

$$L^{\mu} = n_{\ell}u^{\mu} - \sigma_{\ell} P^{\mu\nu}\left(T\partial_{\nu}\frac{\mu_{\ell}}{T} + \ell u^{\lambda}\Xi_{\lambda\nu} - \ell\mu_{\nu}\right)$$

$$\delta p = s\delta T + n\delta\mu + n_{\ell}\delta\mu_{\ell}$$

$$\epsilon = Ts + \mu n + \mu_{\ell}n_{\ell} - p$$

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} + p\, g^{\mu\nu} - \chi\,\mu^{\mu}\mu^{\nu} - 2\eta\, P^{\mu\rho}P^{\nu\sigma}\nabla_{\langle\rho}u_{\sigma\rangle} - \zeta\, P^{\mu\nu}\nabla_{\lambda}u^{\lambda}$$

$$\sigma, \sigma_{\ell}, \eta, \zeta \geq 0$$

- Note that σ_{ℓ} now behaves like a **conductivity** for the defect flux, as well as results in **string charge relaxation**

$$\nabla_{\mu}J^{\mu\nu} = -\ell L^{\nu} \quad \Longrightarrow \quad \underbrace{u^{\mu}\partial_{\mu}n^{\nu} + \ldots}_{-\partial_{\mu}n^{\mu}\,u^{\nu}} = \underbrace{-\ell^{2}\sigma_{\ell}\,\mu^{\nu}(n)}_{-\ell n_{\ell}\,u^{\nu}} + \ell\sigma_{\ell}\,P^{\nu\rho}\partial_{\rho}\mu_{\ell} + \ldots$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu_\mu = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same **fluid sound** and **shear modes**.
- The charge fluctuations give rise to two **diffusive modes**

$$\mu_\perp : \quad \omega = -\,iD_n k^2 - i\Gamma$$

$$\mu_\parallel : \quad \omega = -\,iD_\ell k^2 - i\Gamma$$

$$D_n = \frac{\sigma}{\chi}, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}, \qquad n_\mu = \chi\,\mu_\mu$$

$$D_\ell = \frac{\sigma_\ell}{\chi_\ell}$$

- The $\mu_\parallel$ mode obeys a **damping-attenuation relation**

$$\Gamma = D_\ell\, k_0^2$$

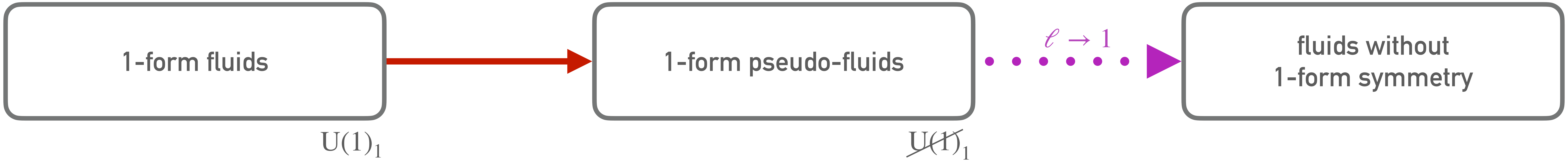
$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2 \chi_\ell}}$$

"Debye length"

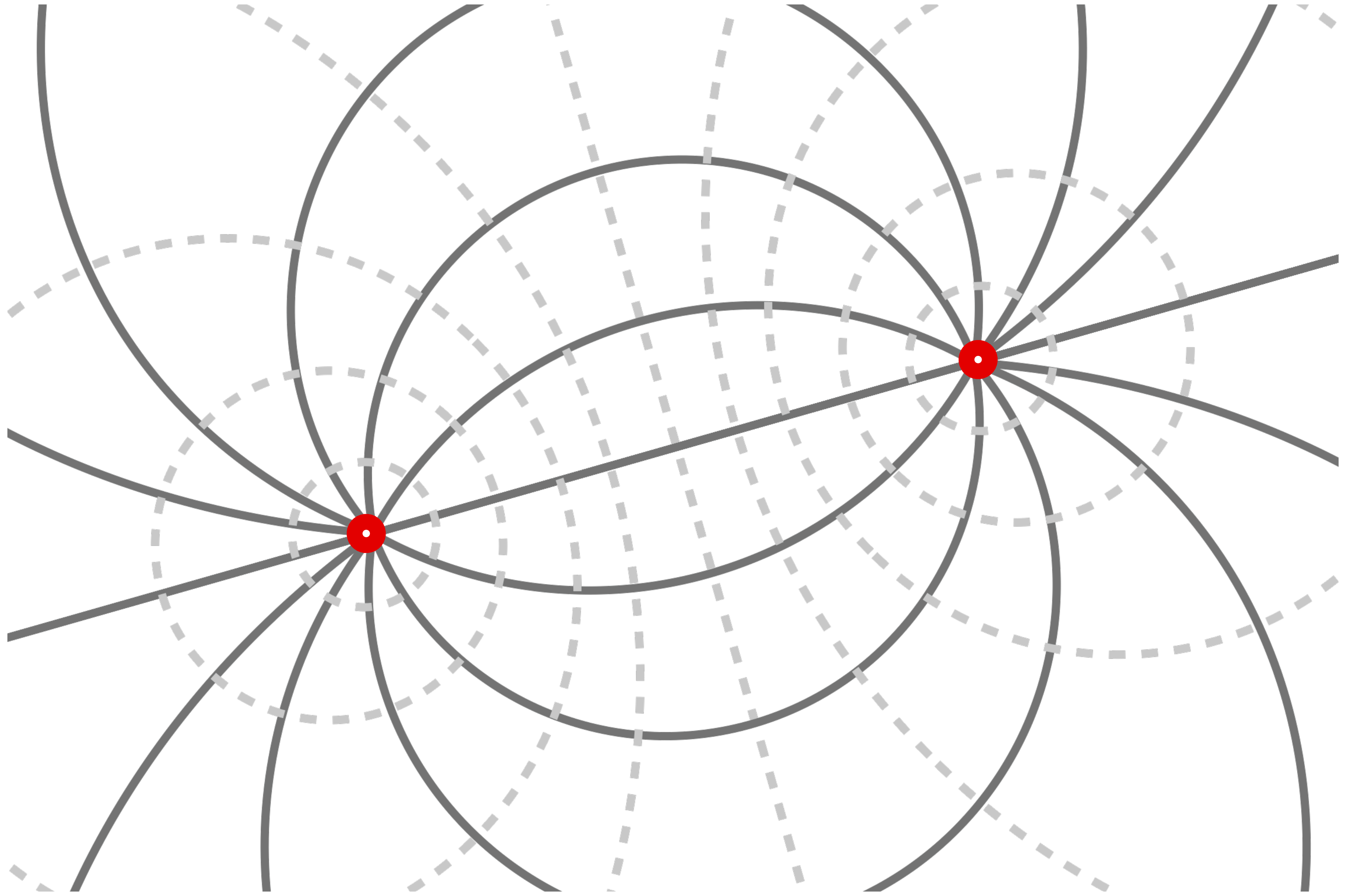
Such relations are a generic feature of systems where a symmetry is **spontaneously+explicitly** broken.

TOPOLOGICAL PHASE TRANSITIONS

- If we increase the strength of defects, i.e. increase ℓ, the charge fluctuations gap out and we are left with a fluid without 1-form symmetry.



$$
\begin{aligned}
u^\mu \partial_\mu n^\nu + \ldots &= -\Gamma n^\nu + \ell D_\ell P^{\nu\rho} \partial_\rho n_\ell + \ldots \\
-\partial_\mu n^\mu u^\nu \quad &\quad -\ell n_\ell u^\nu
\end{aligned}
$$



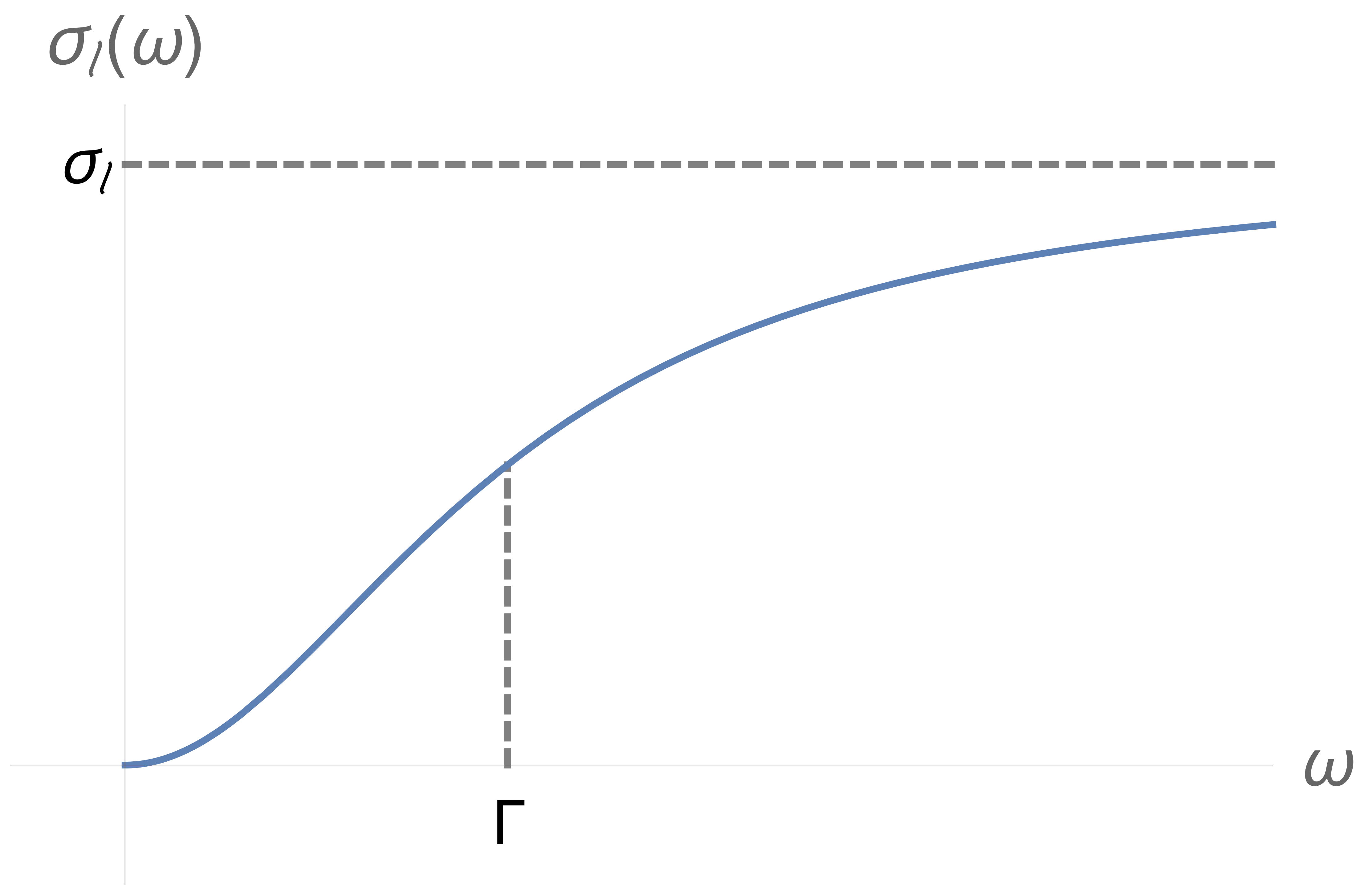
- The same discussion also applies to the phase transition of 0-form superfluids to 0-form fluids, mediated by vortices.

TOPOLOGICAL PHASE TRANSITIONS

- Optical conductivities

$$\sigma(\omega) = \mathrm{Re}\frac{i}{\omega}G^R_{J^{xy}J^{xy}}(\omega) = \sigma$$

$$\sigma_\ell(\omega) = \mathrm{Re}\frac{i}{\omega}G^R_{L^xL^x}(\omega) \ = \sigma_\ell\frac{\omega^2/\Gamma^2}{1+\omega^2/\Gamma^2}$$



HIGHER-FORM SUPERFLUIDS

with approximate higher-form symmetry

1-FORM SUPERFLUIDS IN EQUILIBRIUM

- In the superfluid phase, the higher-form symmetry is **completely spontaneously-broken**

$$\varphi \to \varphi - \Lambda_t \qquad \mu_i = -\partial_i\varphi + A_{ti}$$

$$\phi_i \to \phi_i - \Lambda_i \qquad \xi_{ij} = \partial_i\phi_j - \partial_j\phi_i + A_{ij}$$

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi\mathcal{D}\phi_i \exp\int \mathrm{d}^3x \left(\frac{1}{2}\chi\,\mu_i\mu^i - \frac{1}{4\tilde{\chi}}\xi_{ij}\xi^{ij} + \dots\right)$$

$$\implies \qquad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = -\frac{1}{\tilde{\chi}}\xi^{ij}$$

- The configuration equations imply

$$\partial_i J^{ti} = 0 \implies \partial_i\partial^i\phi = 0$$

$$\partial_t J^{ti} + \partial_k J^{ki} = 0 \implies \partial_k\partial^k\phi^i - \partial^i\partial_k\phi^k = 0$$

Armas, AJ [1811.04913]

1-FORM SUPERFLUIDS IN EQUILIBRIUM

- In the presence of explicit symmetry breaking, the 1-form superfluid can exist in two phases depending on the **0-form defect symmetry** being spontaneously broken or not.
- In the **relaxed/Coulomb phase**, the 0-form defect symmetry is **spontaneously unbroken** and we can only construct the "defect chemical potential"

$$\mu_\ell = -\,\ell\left(\varphi - \Phi_t\right)$$

- In the **pinned/Higgs phase**, the 0-form defect symmetry is **spontaneously broken**

$$\phi_\ell \to \phi_\ell - \Lambda_\ell$$

$$\nabla_\mu J^{\mu\nu} = \ell L^\nu$$

$$\nabla_\mu L^\mu = 0$$

This allows us to also construct the **phase misalignment vector**

$$\psi_i = \ell\left(\phi_i - \Phi_i - \partial_i \phi_\ell\right)$$

In the language of Higgs mechanism, the 1-form phase ϕ_i eats the 0-form phase ϕ_ℓ to become massive.

1-FORM SUPERFLUIDS IN EQUILIBRIUM

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int \mathrm{d}^3x \left(\frac{1}{2}\chi\,\mu_i\mu^i - \frac{1}{4\tilde{\chi}}\xi_{ij}\xi^{ij} + \frac{1}{2}\chi_\ell\,\mu_\ell^2 - \frac{m^2}{2}\psi_i\psi^i + \ldots \right)$$

$$\implies \quad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = -\frac{1}{\tilde{\chi}}\xi^{ij}$$

$$L^t = n_\ell = \chi_\ell\mu_\ell \qquad L^i = m^2\psi^i$$

- Classical configuration equation for φ, ϕ_i impose the conservation equations in equilibrium

$$\partial_i J^{ti} = \ell L^t \implies \partial_i\partial^i\varphi = k_0^2\varphi$$

$$\partial_t J^{ti} + \partial_k J^{ki} = -\ell L^i \implies \partial_k\partial^k\phi^i - \partial^i\partial_k\phi^k = k_{0\phi}^2\phi^i$$

$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2\chi_\ell}} \qquad \frac{1}{k_{0\phi}} = \sqrt{\frac{1}{\ell^2 m^2\tilde{\chi}}}$$

"Debye length" *"London depth"*

1-FORM SUPERFLUID DYNAMICS

- The **conservation equations** for a 1-form superfluid remain the same

$$\nabla_\mu J^{\mu\nu} = \ell L^\nu \qquad \nabla_\mu T^{\mu\nu} = \frac{1}{2} F^{\nu\rho\sigma} J_{\rho\sigma} + \ell \Xi^{\nu\rho} L_\rho$$

$$\nabla_\mu L^\mu = 0$$

$$\frac{u^\mu}{T} = \beta^\mu$$

Hydrodynamic fields are Λ^β_μ, Λ^β_ℓ, β^μ and ϕ_μ, ϕ_ℓ.

$$\beta^\mu \phi_\mu = \varphi$$

- We have a **Josephson equation** for ϕ_μ

$$\xi_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$$

$$\pounds_\beta \phi_\mu = \Lambda^\beta_\mu + \ldots \quad \Longrightarrow \quad u^\mu \xi_{\mu\nu} = \mu_\nu + \ldots$$

$$\frac{\mu_\mu}{T} = \Lambda^\beta_\mu + \beta^\lambda A_{\lambda\mu} - \partial_\mu \varphi$$

- We also have a **Josephson equation** for ϕ_ℓ in the pinned/Higgs phase

$$\frac{\mu_\ell}{T} = -\ell\left(\varphi - \beta^\mu \Phi_\mu - \Lambda^\beta_\ell\right)$$

$$\pounds_\beta \phi_\ell = \Lambda^\beta_\ell + \ldots \quad \Longrightarrow \quad u^\mu \psi_\mu = -\mu_\ell + \ldots$$

$$\psi_\mu = -\ell\left(\phi_\mu - \Phi_\mu\right)$$

GAUGE REDUNDANCY

- There is a **gauge redundancy** in the description that can be obtained by setting

$$\Lambda_\mu = -\partial_\mu \lambda, \qquad \Lambda_\ell = -\lambda$$

which leaves the background fields $A_{\mu\nu}, \Phi_\mu$ invariant.

- The dynamical fields transform as

$$\delta_\lambda \Lambda^\beta_\mu = \partial_\mu \pounds_\beta \lambda \qquad \delta_\lambda \beta^\mu = 0 \qquad \delta_\lambda \phi_\mu = \partial_\mu \lambda$$

$$\delta_\lambda \Lambda^\beta_\ell = \pounds_\beta \lambda \qquad \delta_\lambda \phi_\ell = \lambda$$

The physical hydrodynamic fields $\mu_\mu, \mu_\ell, T, u^\mu, \xi_{\mu\nu}, \psi_\mu$ are invariant under these gauge transformations.

1-FORM SUPERFLUID DYNAMICS

➤ The constitutive relations are given as

$$J^{\mu\nu} = 2u^{[\mu}n^{\nu]} - \zeta^{\mu\nu} - \sigma P^{\mu\rho}P^{\nu\sigma}\left(2T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} + u^{\lambda}F_{\lambda\rho\sigma}\right)$$

$$\zeta^{\mu\nu} = \frac{1}{\tilde{\chi}}P^{\mu\rho}P^{\nu\sigma}\xi_{\rho\sigma}$$

$$L^{\mu} = n_{\ell}u^{\mu} + m^2\bar{\psi}^{\mu} - \sigma_{\ell}P^{\mu\nu}\left(T\partial_{\nu}\frac{\mu_{\ell}}{T} + \ell u^{\lambda}\Xi_{\lambda\nu} - \ell\mu_{\nu}\right) - \gamma P^{\mu\nu}\nabla^{\lambda}\zeta_{\lambda\nu}$$

$$\bar{\psi}^{\mu} = P^{\mu\nu}\psi_{\nu}$$

$$u^{\lambda}\xi_{\lambda\mu} = \mu_{\mu} - \tilde{\sigma}P_{\mu\nu}\nabla_{\lambda}\zeta^{\lambda\nu} - \gamma P_{\mu\nu}\left(T\partial^{\nu}\frac{\mu_{\ell}}{T} + \ell u_{\lambda}\Xi^{\lambda\nu} - \ell\mu^{\nu}\right)$$

$$u^{\mu}\psi_{\mu} = -\mu_{\ell} + \ell\tilde{\sigma}_{\psi}\nabla_{\mu}\left(m^2\bar{\psi}^{\mu}\right)$$

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} + p\,g^{\mu\nu} - \chi\,\mu^{\mu}\mu^{\nu} + \tilde{\chi}\,\zeta^{\mu\rho}\zeta^{\nu}{}_{\rho} - 2\eta\,P^{\mu\rho}P^{\nu\sigma}\nabla_{\langle\rho}u_{\sigma\rangle} - \zeta\,P^{\mu\nu}\nabla_{\lambda}u^{\lambda}$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu_\mu = \xi_{\mu\nu} = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same **fluid sound** and **shear modes**.
- The transverse charge and Goldstone fluctuations give rise to the **screened photon mode** in the Coulomb/relaxed phase

$$\mu_\perp, \phi_\perp : \quad \left(i\omega - D_n k^2 - \Gamma\right)\left(i\omega - \tilde{D}_n k^2\right) + v_\perp^2 k^2 = 0$$

$$v_\perp^2 = \frac{\lambda^2}{\chi \tilde{\chi}}$$

- In the pinned/Higgs phase, you get additional **pinning** and **screening** effects

$$\mu_\perp, \phi_\perp : \quad \left(i\omega - D_n k^2 - \Gamma\right)\left(i\omega - \tilde{D}_n k^2 - \tilde{\Omega}\right) + \omega_0^2 + v_\perp^2 k^2 = 0$$

$$\omega_0^2 = v_\perp^2 k_{0\phi}^2$$

$$\tilde{\Omega} = \tilde{D}_n k_{0\phi}^2$$

$$\frac{1}{k_{0\phi}} = \sqrt{\frac{1}{\ell^2 m^2 \tilde{\chi}}}$$

"London depth"

- Similar results hold for q-form superfluids with vortices or impurities for any q.

LINEARISED FLUCTUATIONS

- The longitudinal charge mode in the relaxed/Coulomb phase returns the previous **damped diffusive mode**

$$\mu_{\parallel} : \quad i\omega = D_{\ell} k^2 + \Gamma$$

- For pinned/Higgs phase, this couples to longitudinal Goldstone fluctuation and gives a **second sound mode**

$$\mu_{\parallel}, \phi_{\parallel} : \quad \left(i\omega - D_{\ell} k^2 - \Gamma\right)\left(i\omega - \tilde{D}_{\psi} k^2 - \tilde{\Omega}\right) + \omega_0^2 + v_{\parallel}^2 k^2 = 0$$

$$\Gamma = D_{\ell} k_0^2$$

$$v_{\parallel}^2 = \frac{\lambda^2 m^2}{\chi_{\ell}}$$

$$\omega_0^2 = v_{\parallel}^2 k_0^2$$

$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2 \chi_{\ell}}}$$

"Debye length"

TOPOLOGICAL PHASE TRANSITIONS

- We can implement **topological phase transitions** by increasing the strength of ℓ, thereby increasing the strength of explicit symmetry breaking.
- The product of the phase transition depends on if we are starting form the relaxed/Coulomb phase of pinned/Higgs phase.
- In the relaxed/Coulomb phase, we arrive at a **$(d-2)$-form fluid**. In the context of electromagnetism, this describes **magnetohydrodynamics** with conserved magnetic field lines.

$$\omega = -i\left(\tilde{D}_n + \frac{v_\perp^2}{\Gamma}\right)k^2 = -\frac{i}{\tilde{\chi}}\left(\tilde{\sigma} + \frac{\lambda^2}{\sigma_\ell}\right)k^2$$

- In the pinned/Higgs phase, we arrive at a **neutral fluid**. In the context of electromagnetism, this describes the Meissner effect, i.e. expulsion of all electromagnetic fields inside a superconductor.

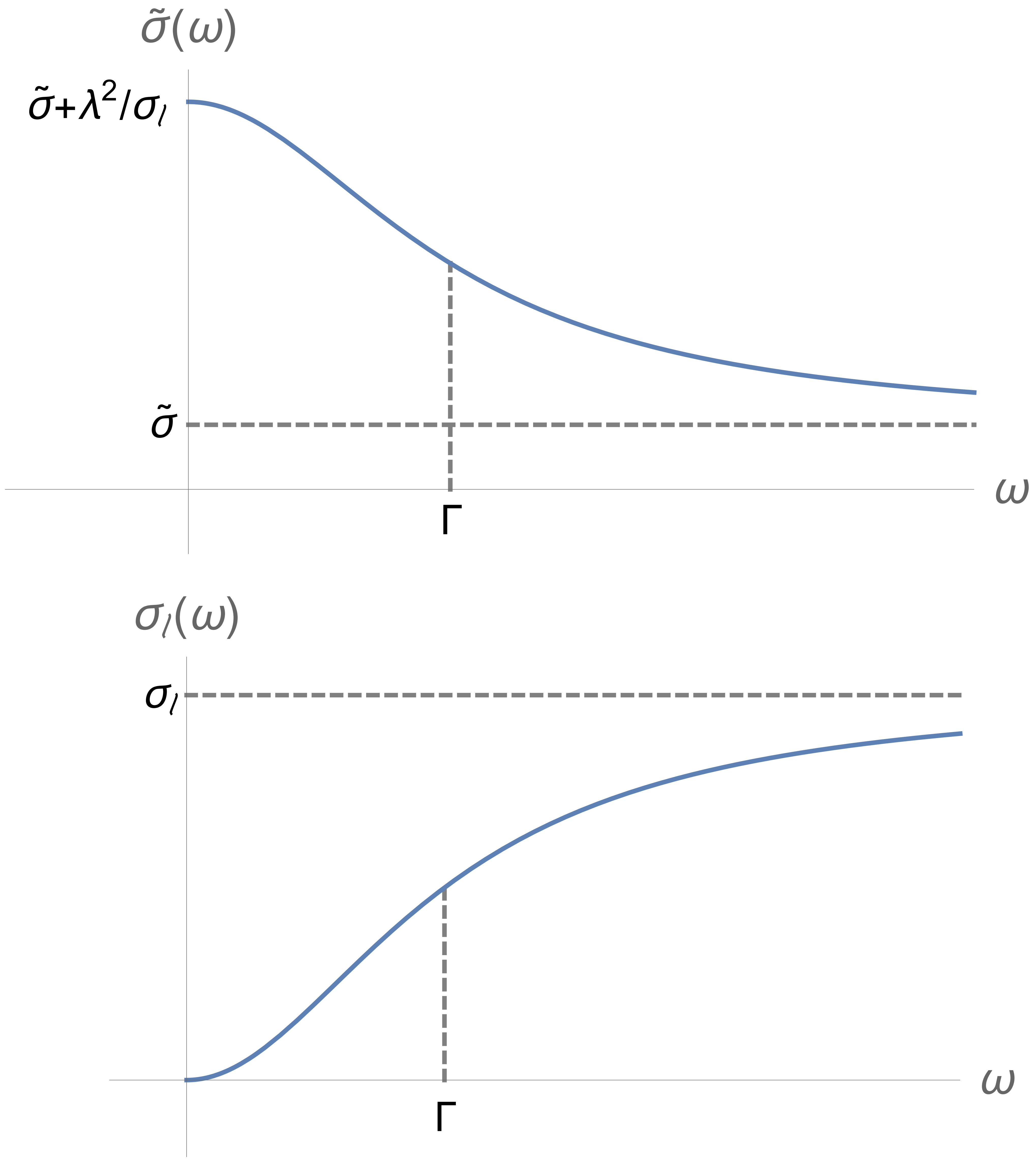
TOPOLOGICAL PHASE TRANSITIONS

- Optical conductivities (Coulomb/relaxed phase)

$$\sigma(\omega) = \mathrm{Re}\frac{i}{\omega}G^R_{J^{xy}J^{xy}}(\omega) = \sigma$$

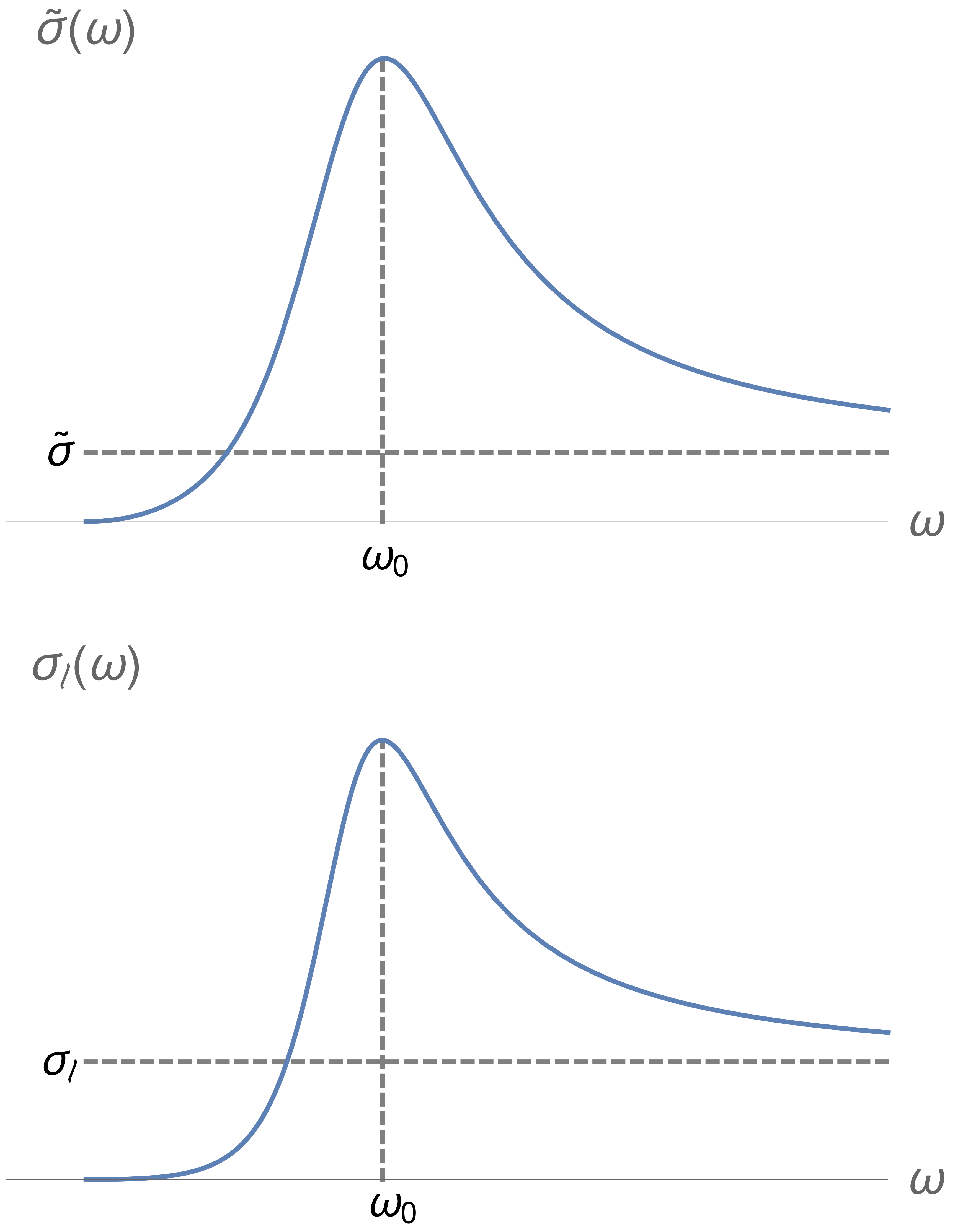
$$\tilde{\sigma}(\omega) = \mathrm{Re}\frac{i}{\omega}G^R_{\xi^{tx}\xi^{tx}}(\omega) = \tilde{\sigma} + \frac{\lambda^2}{\sigma_\ell}\frac{1}{1+\omega^2/\Gamma^2}$$

$$\sigma_\ell(\omega) = \mathrm{Re}\frac{i}{\omega}G^R_{L^xL^x}(\omega) = \sigma_\ell\frac{\omega^2/\Gamma^2}{1+\omega^2/\Gamma^2}$$



TOPOLOGICAL PHASE TRANSITIONS

- Optical correlation functions (Higgs/pinned phase)



HIGHER-FORM SUPERFLUIDS

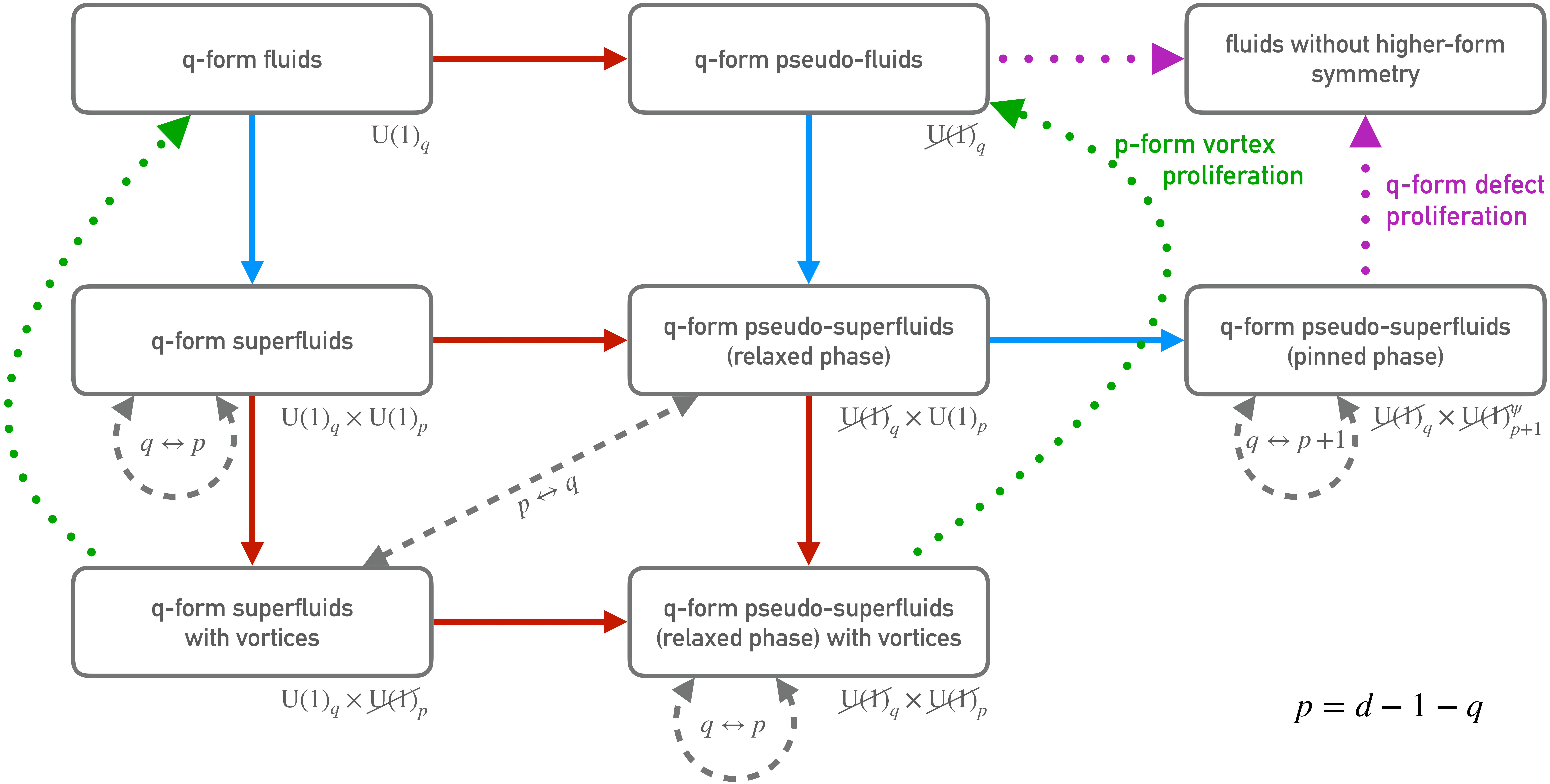
- It is possible to also keep the "magnetic" $(d-2)$-form symmetry of a 1-form superfluid manifest by coupling the system to a $(d-1)$-form gauge field and accounting for the mixed anomaly.
- Explicit breaking of the "magnetic" $(d-2)$-form symmetry give rise to vortices in the 1-form superfluid, realising **magnetic monopoles**. Only relaxed/Coulomb phase can admit vortices.
- Extension to q-form superfluids is straight-forward.

OUTLOOK

© Mildred Thompson "Magnetic Fields"

OUTLOOK

- **Higher-form symmetries** can be used to classify phases of matter with topological order.
- Breaking of continuous higher-form symmetries is associated with **topological defects**, which mediate **topological phase transitions**.
- A hydrodynamic theory with approximate higher-form symmetries provides a model for dynamical phase transitions based on symmetries.
- Further applications include emergent magnetic monopoles in spin ice, plasma phase transitions, melting phase transition in higher-dimensions, superfluid and superconductor phase transitions.



q-form fluids
q-form pseudo-fluids
fluids without higher-form symmetry
$U(1)_q$
$\not{U(1)}_q$
p-form vortex proliferation
q-form defect proliferation
q-form superfluids
q-form pseudo-superfluids (relaxed phase)
q-form pseudo-superfluids (pinned phase)
$U(1)_q \times U(1)_p$
$\not{U(1)}_q \times U(1)_p$
$\not{U(1)}_q \times \not{U(1)}^{\psi}_{p+1}$
$q \leftrightarrow p$
$p \leftrightarrow q$
$q \leftrightarrow p+1$
q-form superfluids with vortices
q-form pseudo-superfluids (relaxed phase) with vortices
$U(1)_q \times \not{U(1)}_p$
$\not{U(1)}_q \times \not{U(1)}_p$
$q \leftrightarrow p$
$p = d - 1 - q$