

HIGHER-FORM SYMMETRIES AND TOPOLOGICAL PHASE TRANSITIONS

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[2301.09628] Armas, AJ

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MOTIVATION

- **Symmetries** are powerful guiding principle for developing effective theories for physical systems without a detailed understanding of their microscopic constituents.
- Equilibrium phases of matter can be organised according to their symmetries and whether these are **spontaneously broken** or **unbroken** in the ground state, commonly known as the **Landau paradigm**.
- Symmetries can even be useful when they are only **approximately** respected by the system.
The canonical example comes from *chiral perturbation theory*, where pions are seen as pseudo-Goldstones of approximate $SU(2)$ chiral symmetry.
Other examples: *pinned crystals*, *pinned charge density waves*, *pseudo-superfluids* etc.



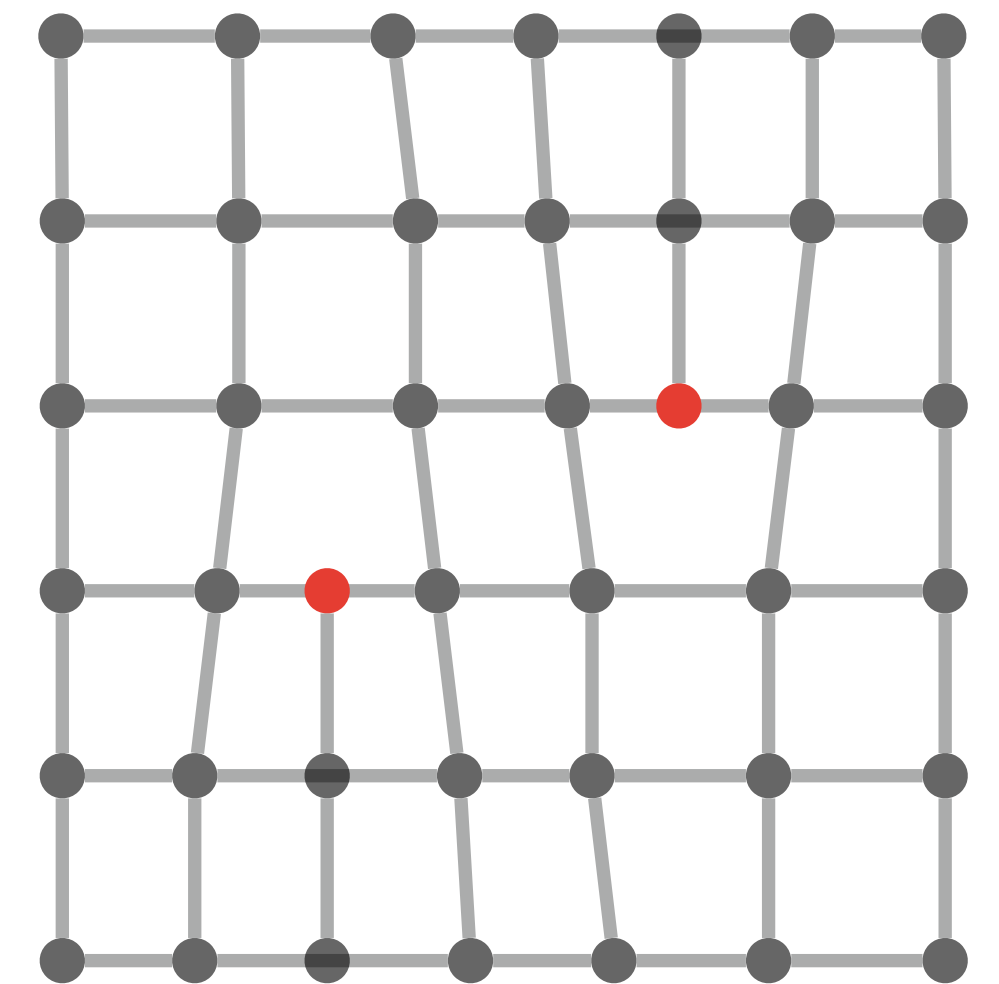
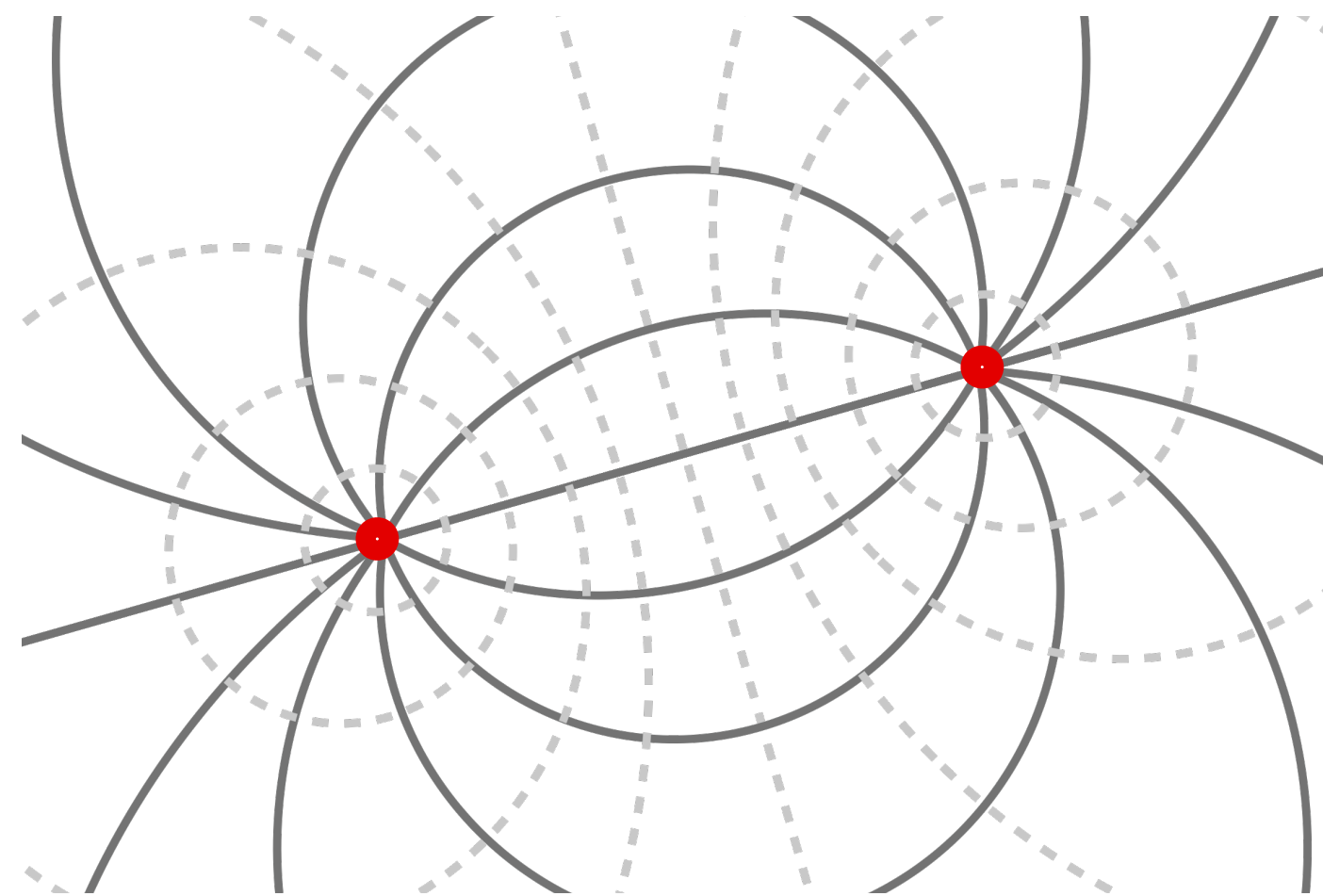
MOTIVATION

- In recent years, the notion of symmetries has been **generalised** to include higher-form symmetries, higher-group symmetries, subsystem symmetries, and non-invertible symmetries.
- These allow for a **generalised Landau paradigm**, that also include exotic phases of matter, such as *topologically ordered states*, *spin liquids*, *fractons*, *topological insulators*, etc.
- The focus of this talk is **continuous higher-form symmetries**, which concerns higher-dimensional charged objects, such as strings and surfaces.
- These describe **topological order** in many-body systems, such as *equipotential planes* in a superfluid, *lattice planes* in a crystal, *magnetic fields* in a plasma, or *electric fields* in a dielectric fluid.



MOTIVATION

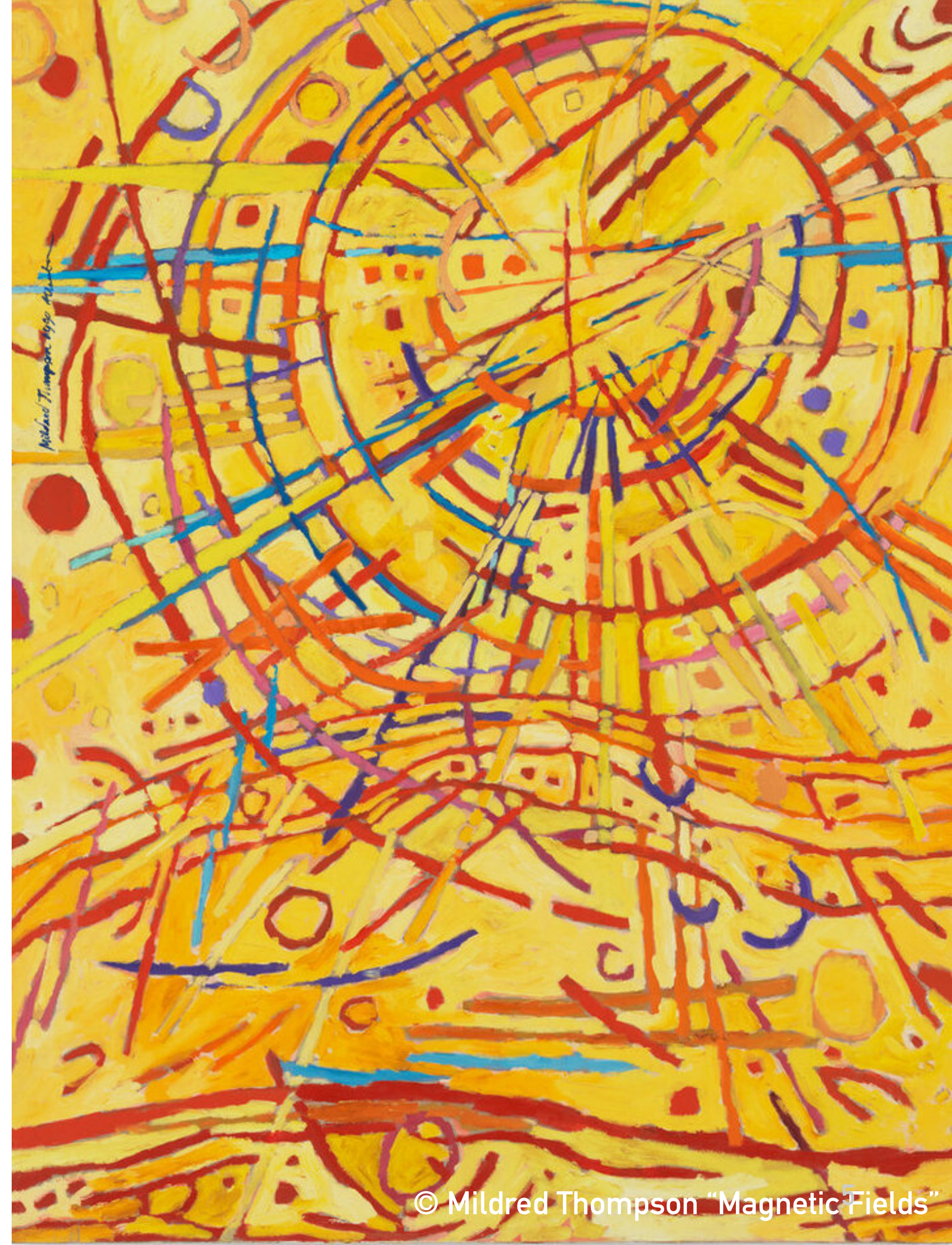
- Explicit breaking of higher-form symmetries describes **topological defects**, such as *superfluid vortices*, *crystal dislocations*, *magnetic monopoles*, or *free charges*.
- Topological defects mediate **topological phase transitions**,¹ wherein a spontaneously broken symmetry gets restored. Examples include *superfluid phase transition*, *melting*, and *plasma phase transition*.



¹Not to be confused with phase transitions between topologically ordered phases.

HIGHER-FORM SYMMETRIES

and their breaking

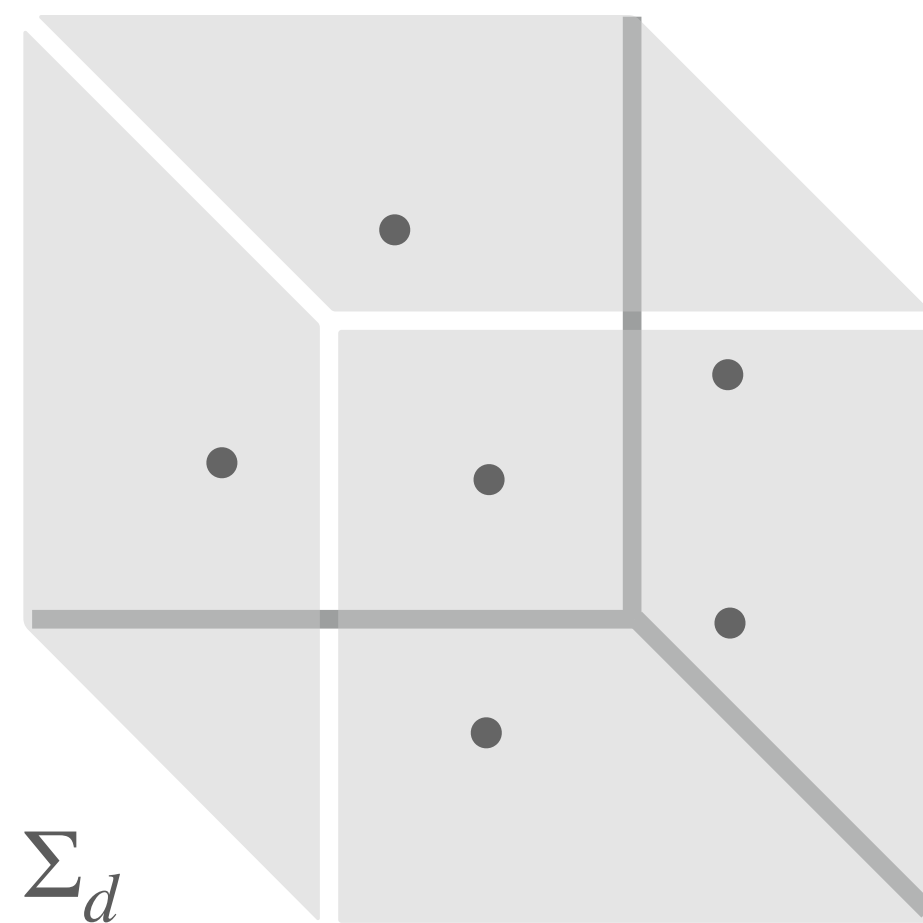


HIGHER-FORM SYMMETRIES

- ▶ Continuous 0-form symmetry:

$$\partial_\mu J^\mu = 0 \quad \implies \quad \partial_t J^t + \partial_i J^i = 0$$

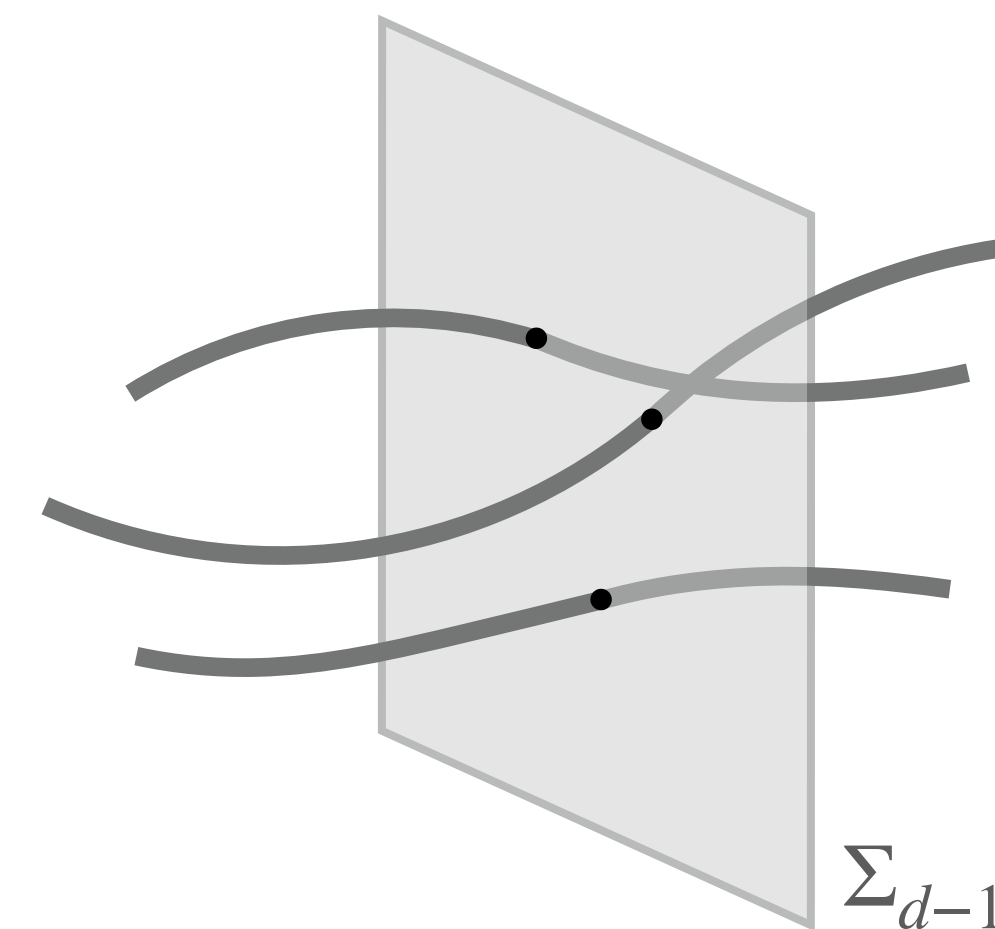
- ▶ The number of charged particles in a volume Σ_d is conserved in **time**.



- ▶ Continuous 1-form symmetry:

$$\partial_\mu J^{\mu\nu} = 0 \quad \implies \quad \begin{aligned} \partial_t J^{ti} + \partial_k J^{ki} &= 0 \\ \partial_i J^{ti} &= 0 \end{aligned}$$

- ▶ The number of charged “strings” passing a cross section Σ_{d-1} are conserved in **time** and under **spatial deformations** of Σ_{d-1} .

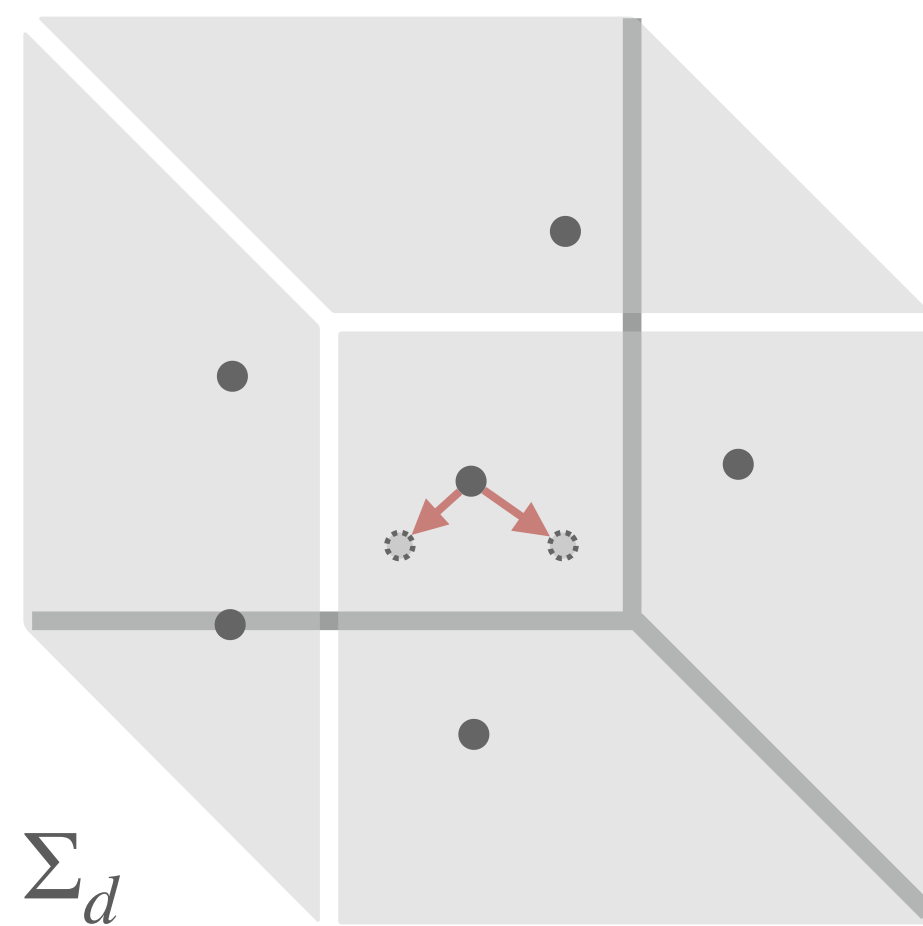


APPROXIMATE HIGHER-FORM SYMMETRIES

- ▶ Continuous approximate 0-form symmetry:

$$\partial_\mu J^\mu = -\ell L$$

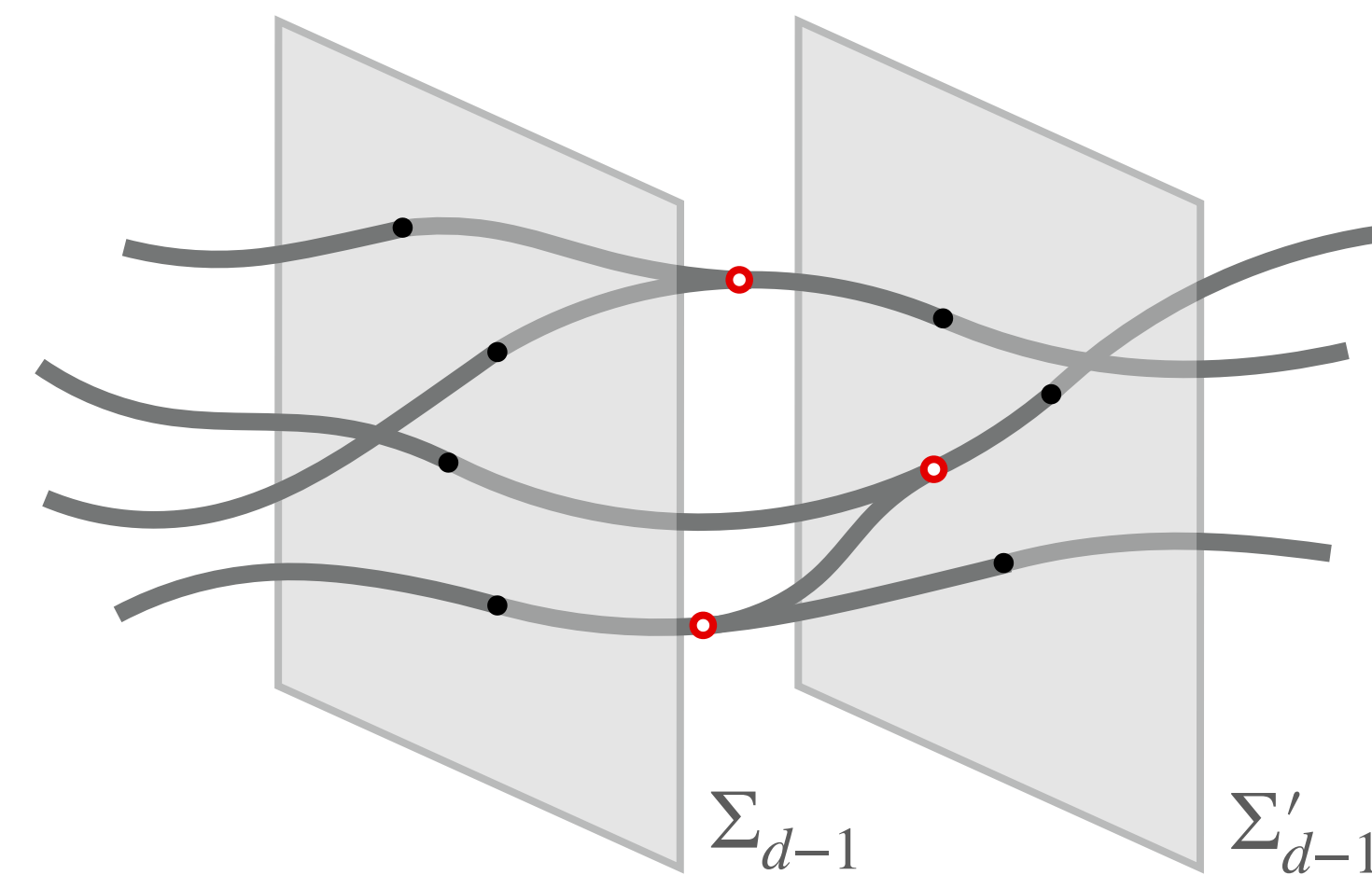
- ▶ Charged particles in a volume Σ_d can be **created/annihilated** in time.



- ▶ Continuous approximate 1-form symmetry:

$$\partial_\mu J^{\mu\nu} = -\ell L^\nu$$

- ▶ Charged “strings” passing a cross section Σ_{d-1} can be **created/annihilated** in time and under spatial deformations of Σ_{d-1} .

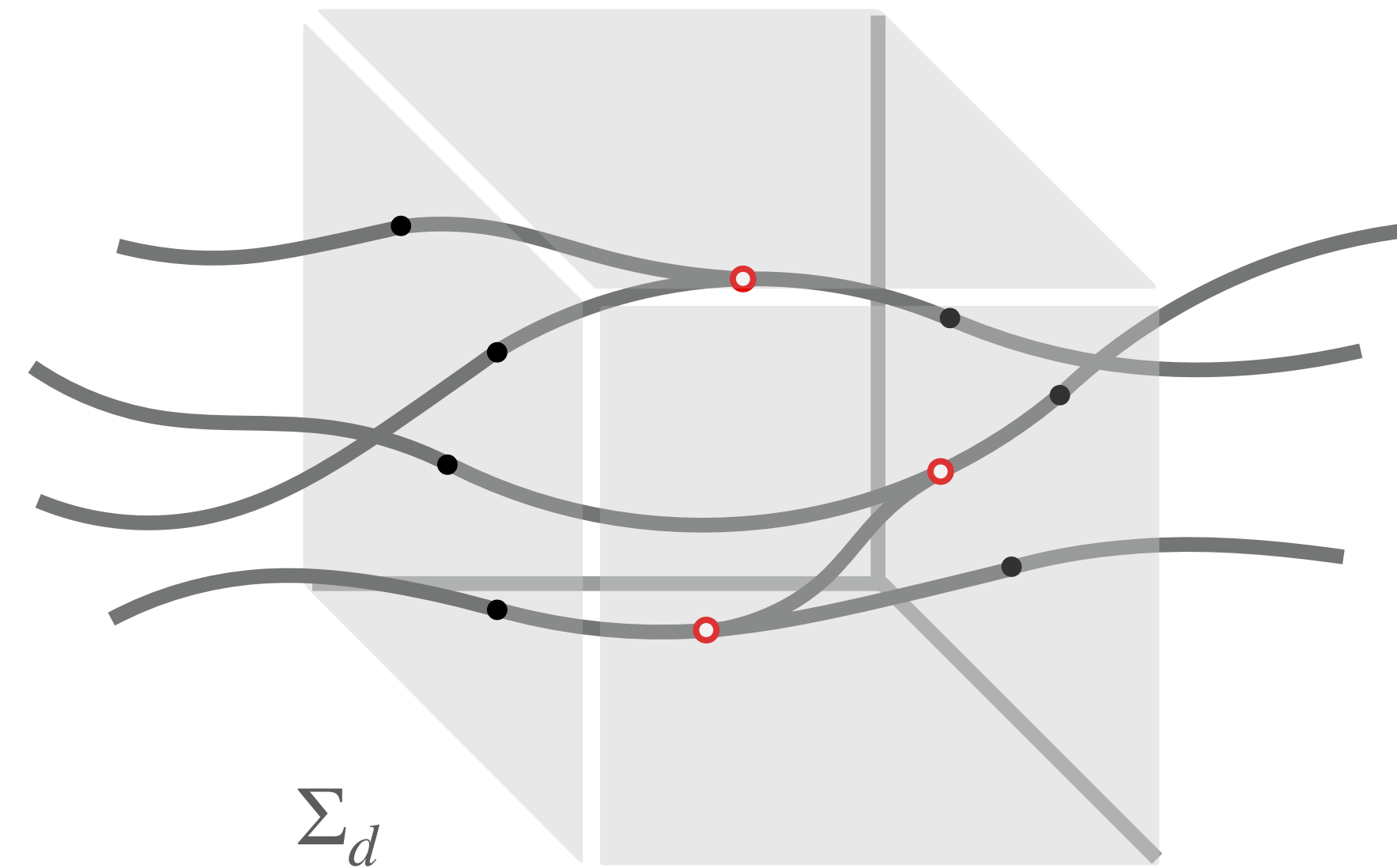


APPROXIMATE HIGHER-FORM SYMMETRIES

- ▶ The defects of a 1-form symmetry themselves furnish an emergent 0-form “defect” symmetry

$$\partial_\mu L^\mu = 0$$

- ▶ The number of **defects** in a volume Σ_d is conserved in **time**.



BACKGROUND FIELDS

- ▶ We can introduce a 1-form gauge field A_μ and a background phase Φ to probe an approximate 0-form symmetry

$$\delta S[A, \Phi] = \int d^d x \left(J^\mu \delta A_\mu + \ell L \delta \Phi \right)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

$$\Phi \rightarrow \Phi - \Lambda$$

- ▶ Similarly, we can introduce a 2-form gauge field $A_{\mu\nu}$ and a background phase Φ_μ to probe an approximate 1-form symmetry

$$\delta S[A, \Phi] = \int d^d x \left(\frac{1}{2} J^{\mu\nu} \delta A_{\mu\nu} + \ell L^\mu \delta \Phi_\mu \right)$$

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

$$\Phi_\mu \rightarrow \Phi_\mu - \Lambda_\mu + \partial_\mu \Lambda_\ell$$

$$\Lambda_\mu \rightarrow -\partial_\mu \Lambda_\ell$$

EXAMPLE: ELECTROMAGNETISM

- ▶ $d = 3$ electromagnetism in vacuum

$$S = - \int d^4x \left(\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} \right) \quad \Rightarrow \quad \begin{aligned} \partial_\mu \mathcal{F}^{\mu\nu} &= 0 \\ \partial_\mu \star \mathcal{F}^{\mu\nu} &= 0 \end{aligned}$$
$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$$

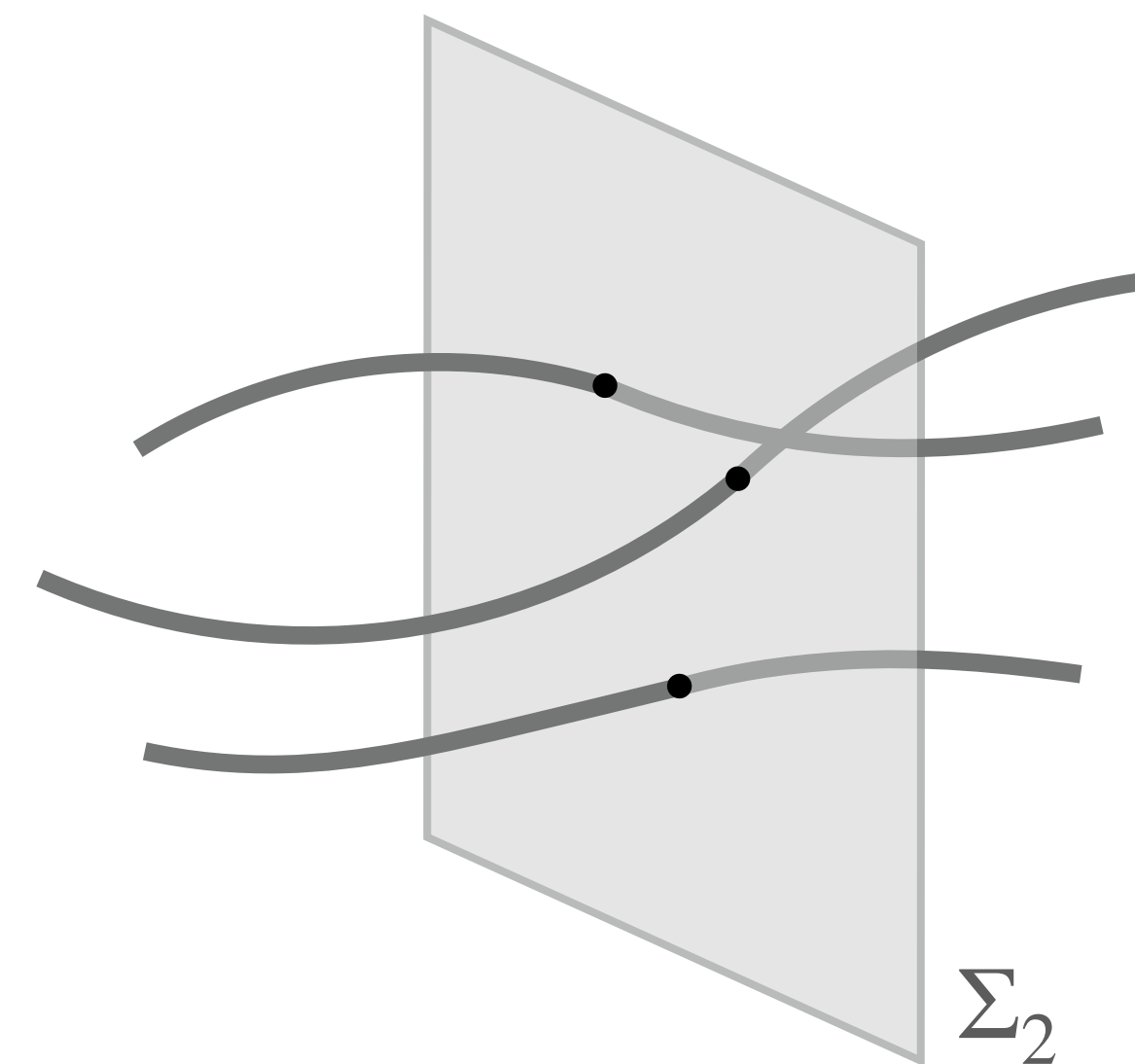
- ▶ It has two 1-form symmetries:

$$J^{\mu\nu} = - \mathcal{F}^{\mu\nu} \quad \tilde{J}^{\mu\nu} = \star \mathcal{F}^{\mu\nu}$$

The associated charged objects are **electric** and **magnetic field lines**.

- ▶ These symmetries also persist in the presence of polarised/dielectric matter

$$J^{\mu\nu} = - \mathcal{F}^{\mu\nu} + \mathcal{M}^{\mu\nu}$$



EXAMPLE: ELECTROMAGNETISM

- In the presence of **free electric charges**, the electric 1-form symmetry is **explicitly broken**, but the magnetic 1-form symmetry persists.

$$S = - \int d^4x \left(\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \mathbf{D}_\mu \Psi^* \mathbf{D}^\mu \Psi + V(\Psi^* \Psi) \right)$$

$$D_\mu = \partial_\mu - i\ell \mathcal{A}_\mu$$

$$J^{\mu\nu} = - \mathcal{F}^{\mu\nu}$$

$$\tilde{J}^{\mu\nu} = \star \mathcal{F}^{\mu\nu}$$

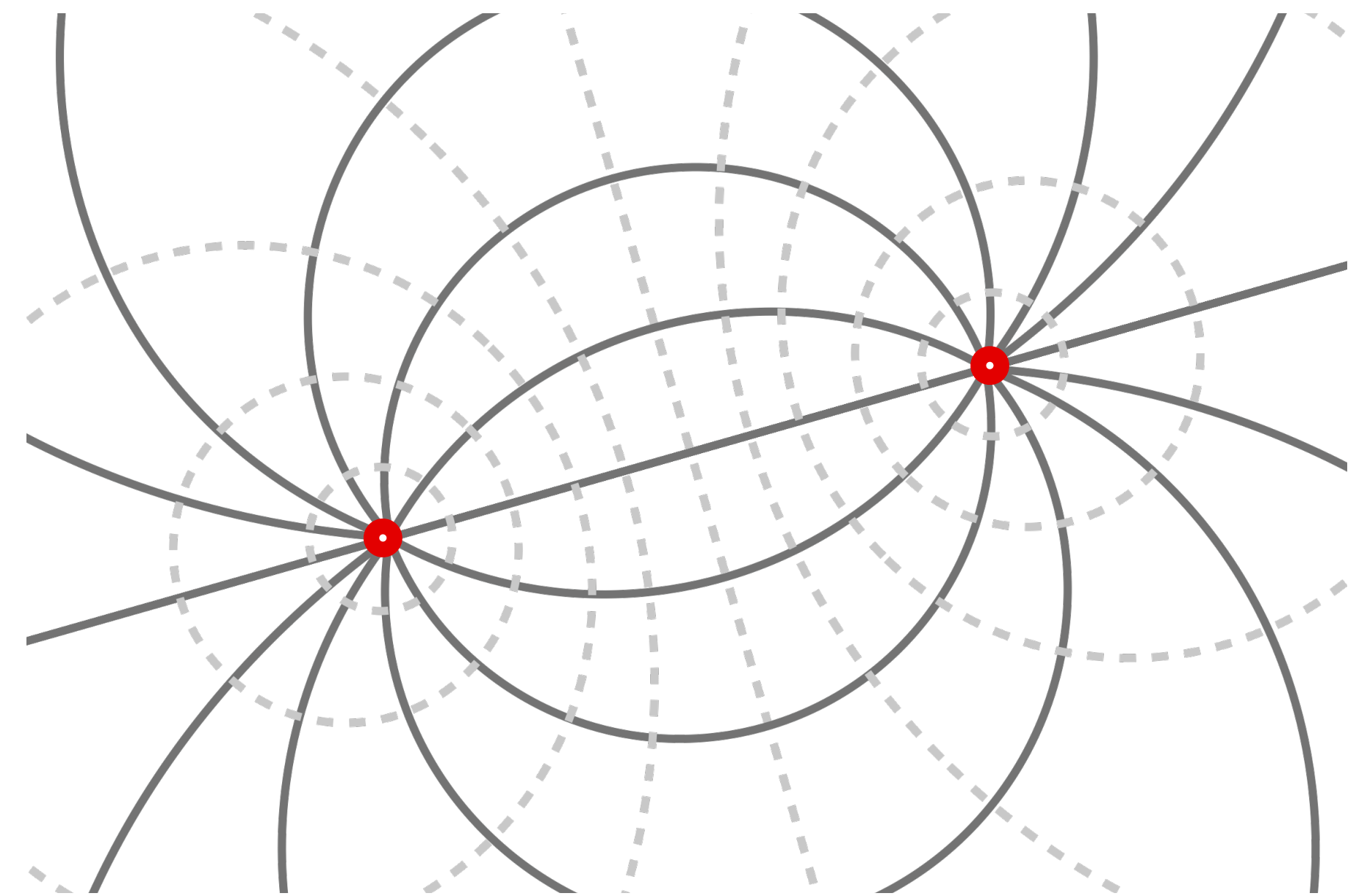
$$\implies$$

$$\partial_\mu J^{\mu\nu} = -\ell L^\nu$$

$$\partial_\mu \tilde{J}^{\mu\nu} = 0$$

$$L^\mu = i (\Psi^* D^\mu \Psi - D^\mu \Psi^* \Psi)$$

- Similarly, breaking of the magnetic 1-form symmetry amounts to the introduction of **magnetic monopoles**.



EXAMPLE: ELECTROMAGNETISM

- We can manifest the electric 1-form symmetry via

$$S[A, \Phi] = - \int d^4x \left(\frac{1}{4} \xi^{\mu\nu} \xi_{\mu\nu} + \mathbf{D}_\mu \Psi^* \mathbf{D}^\mu \Psi + V(\Psi^* \Psi) \right)$$

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \quad \mathcal{A}_\mu \rightarrow \mathcal{A}_\mu - \Lambda_\mu$$

$$\Phi_\mu \rightarrow \Phi_\mu - \Lambda_\mu + \partial_\mu \Lambda_\ell \quad \Psi \rightarrow e^{-i\ell \Lambda_\ell} \Psi$$

$$\xi_{\mu\nu} = \mathcal{F}_{\mu\nu} + A_{\mu\nu}$$

$$\psi_\mu = \ell \left(\mathcal{A}_\mu - \Phi_\mu \right)$$

$$D_\mu = \partial_\mu - i\psi_\mu$$

Electromagnetism is **1-form superfluidity**: 1-form symmetry is **spontaneously** broken. Free electric charges arise as **explicit breaking** of the 1-form symmetry.

- The “local” U(1) symmetry of electromagnetism: $\Lambda_\mu = -\partial_\mu \lambda$, $\Lambda_\ell = -\lambda$.
- It is also possible to manifest the magnetic 1-form symmetry.¹

¹Both higher-form symmetries cannot be gauged together on account of a mixed 't Hooft anomaly.

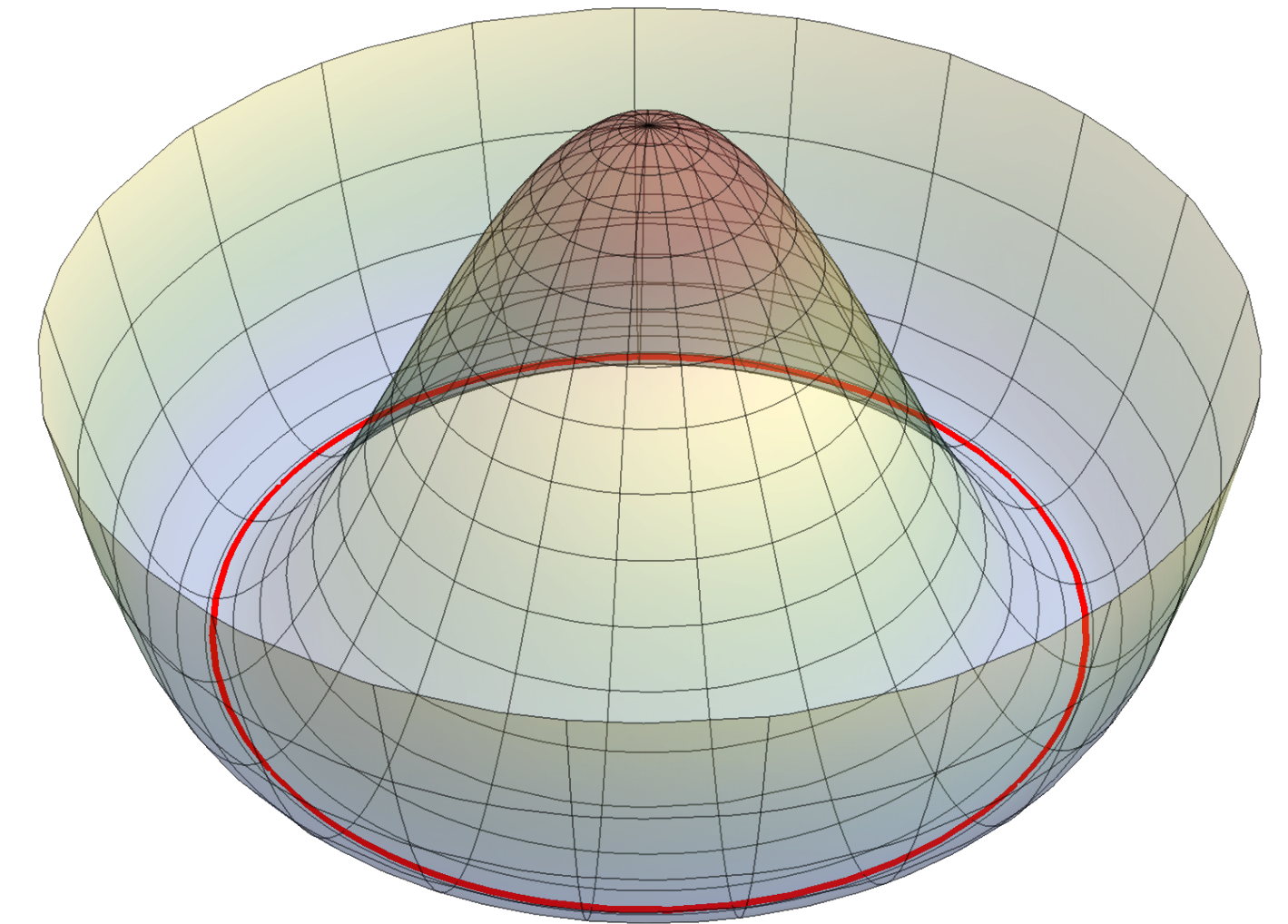
EXAMPLE: SUPERCONDUCTIVITY

- In the **superconducting** phase, Ψ condenses around a nonzero vev $|\Psi| = m/\sqrt{2}$.
- The phase fluctuations around the vev are described by the **Ginzburg-Landau theory**

$$S[A, \Phi] = - \int d^4x \left(\frac{1}{4} \xi^{\mu\nu} \xi_{\mu\nu} + \frac{1}{2} m^2 \psi_\mu \psi^\mu + \dots \right)$$

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \quad \mathcal{A}_\mu \rightarrow \mathcal{A}_\mu - \Lambda_\mu$$

$$\Phi_\mu \rightarrow \Phi_\mu - \Lambda_\mu + \partial_\mu \Lambda_\ell \quad \phi_\ell \rightarrow \phi_\ell - \Lambda_\ell$$



- In the superconducting (Higgs) phase, the 1-form symmetry is **spontaneously** and weakly **explicitly** broken. The emergent 0-form “defect” symmetry is also **spontaneously** broken.

$$\Psi = \frac{m}{\sqrt{2}} e^{i\ell\phi_\ell}$$

$$\psi_\mu = \ell \left(\mathcal{A}_\mu - \Phi_\mu - \partial_\mu \phi_\ell \right)$$



HIGHER-FORM SYMMETRIES

- In general spatial dimensions d , electromagnetism has an **electric 1-form** and **magnetic $(d - 2)$ -form** symmetry. The defects are free electric charges and magnetic monopoles.
- Electromagnetism can be viewed as a 1-form or $(d - 2)$ -form superfluid.¹
- Ordinary 0-form superfluids have a $(d - 1)$ -form symmetry, with the defects being **vortices**.²

$$J^{\mu\nu\dots} = \epsilon^{\lambda\mu\nu\dots} \partial_\lambda \phi$$

- Crystals also have a $(d - 1)$ -form symmetry, with the defects being **dislocations**.³

$$J^I \mu\nu\dots = \epsilon^{\lambda\mu\nu\dots} \partial_\lambda \phi^I$$

¹Hofman, Iqbal [1802.09512]; Armas, AJ [1808.01939, 1811.04913]

²Delacrétaz, Hofman, Mathys [1908.06977]

³Grozdanov, Poovuttikul [1801.03199]; Armas, AJ [1908.01175];
Armas, Heumen, AJ, Lier [2211.02117]

THERMAL EQUILIBRIUM

with approximate higher-form
symmetry





THERMAL EQUILIBRIUM

- Many-body systems at thermal equilibrium can be characterised by their **thermal partition function**.
- Thermal partition function is a functional of background fields and can be used to obtain equilibrium values of (approximately) conserved densities and fluxes.
- For systems with **spontaneously unbroken** symmetries, the thermal partition function is a “local” functional of background fields.
- For systems with **spontaneously broken** symmetries, the thermal partition function is “non-local”, and is given by a functional integral over the time-independent configurations of the **Goldstone fields**.

0-FORM HYDROSTATICS

- For a 0-form symmetry, a thermal ensemble with constant charge density is described by a **thermal partition function**

$$\mathcal{Z}[A] = \exp \int d^d x \left(\frac{1}{2} \chi \mu^2 + \dots \right) \quad \mu = \mu_0 + A_t$$

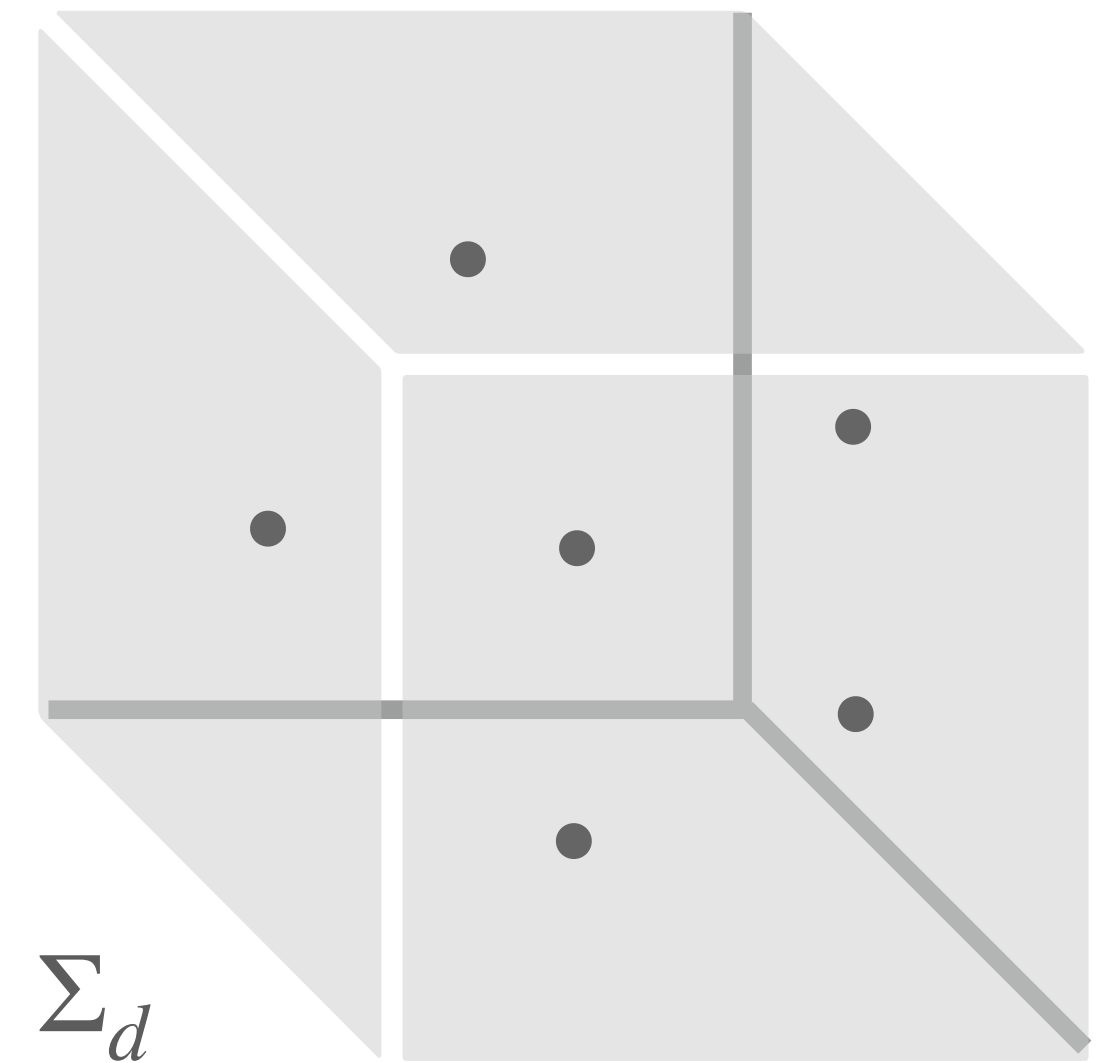
$$\implies \quad J^t = n = \chi \mu \quad J^i = 0$$

- This works because A_t is invariant under **time-independent gauge transformations**

$$A_t \rightarrow A_t + \partial_t \Lambda$$

This is no longer true for higher-form symmetries

$$A_{ti} \rightarrow A_{ti} + \partial_t \Lambda_i - \partial_i \Lambda_t$$



0-FORM SUPERFLUID HYDROSTATICS

- ▶ When the 0-form symmetry is spontaneously broken, the theory admits a **Goldstone field** ϕ , which can be used to define

$$\begin{aligned} \phi &\rightarrow \phi - \Lambda & \mu &= \mu_0 + A_t \\ & & \xi_i &= \partial_i \phi + A_i \end{aligned}$$

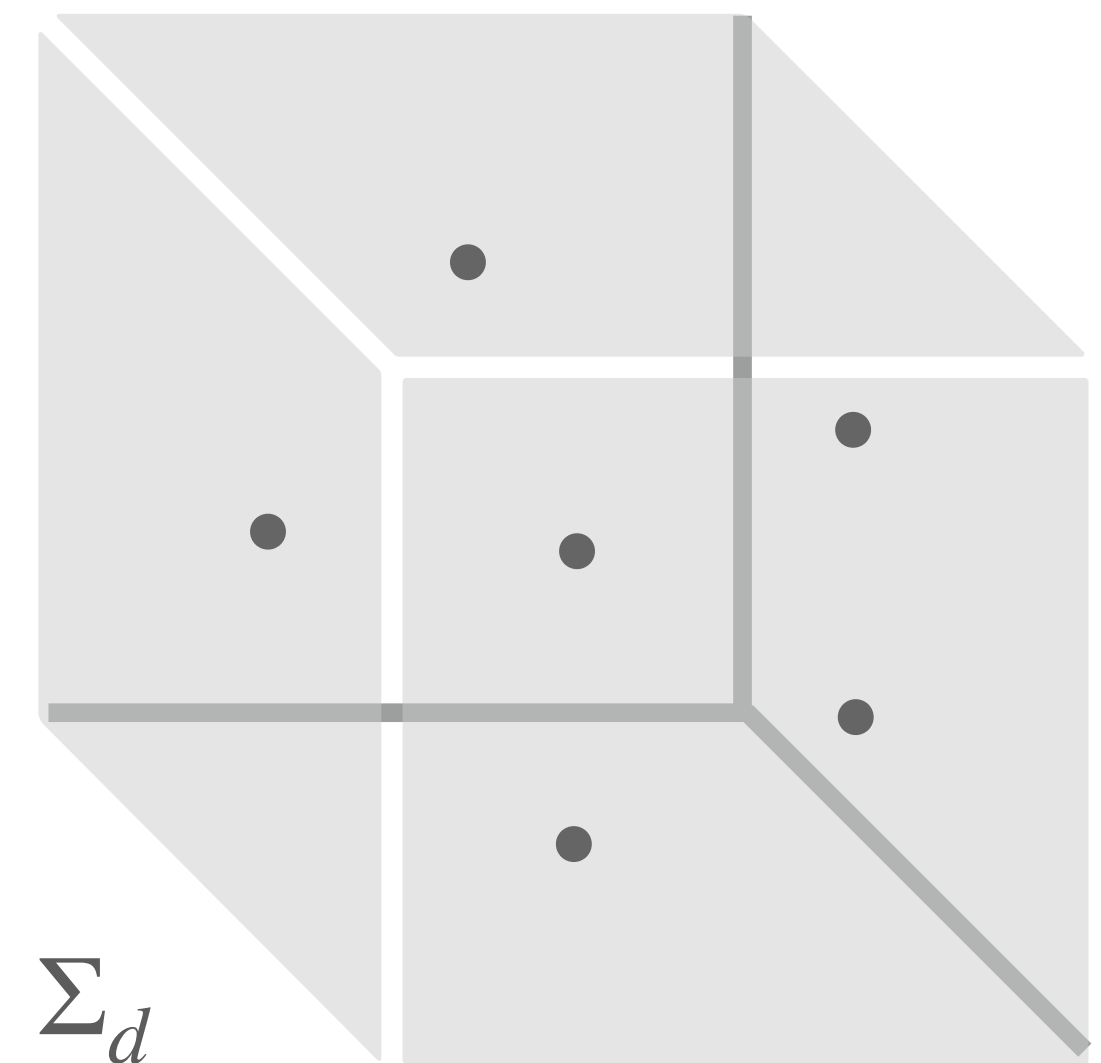
- ▶ The **thermal partition function** is now given by

$$\mathcal{Z}[A] = \int \mathcal{D}\phi \exp \int d^d x \left(\frac{1}{2} \chi \mu^2 - \frac{1}{2} f_s \xi_i \xi^i + \dots \right)$$

$$\implies \quad J^t = n = \chi \mu \quad J^i = -f_s \xi^i$$

- ▶ Classical configurations are given by solutions of

$$\partial_t J^t + \partial_i J^i = 0 \implies \partial_k \partial^k \phi = 0$$



1-FORM HYDROSTATICS

- To obtain non-trivial thermodynamic states with 1-form symmetry, we need to **partially-spontaneously break** the 1-form symmetry in the time-direction

$$\varphi \rightarrow \varphi - \Lambda_t \quad \mu_i = -\partial_i \varphi + A_{ti}$$

This allows us to construct a **thermal partition function**

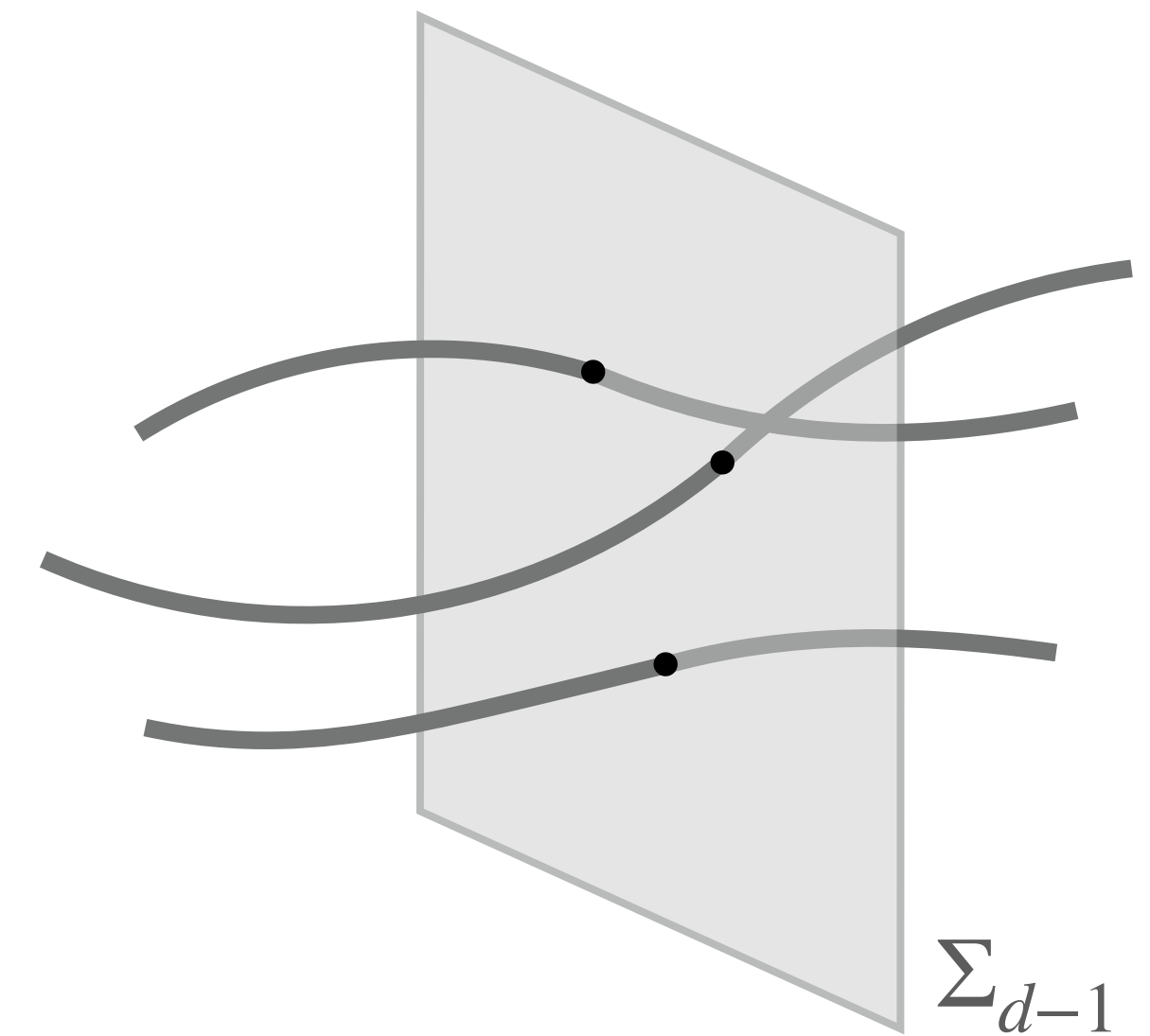
$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int d^d x \left(\frac{1}{2} \chi \mu_i \mu^i + \dots \right)$$

$$\implies J^{ti} = n^i = \chi \mu^i \quad J^{ij} = 0$$

e.g. free electrostatics/magnetostatics.

- Classical configuration equation of φ implements the **Gauss constraint**

$$\partial_i J^{ti} = 0 \implies \partial_i \partial^i \varphi = 0$$



1-FORM SUPERFLUID HYDROSTATICS

- In the superfluid phase, the 1-form symmetry is **completely spontaneously-broken**

$$\varphi \rightarrow \varphi - \Lambda_t \qquad \mu_i = -\partial_i \varphi + A_{ti}$$

$$\phi_i \rightarrow \phi_i - \Lambda_i \qquad \xi_{ij} = \partial_i \phi_j - \partial_j \phi_i + A_{ij}$$

- The partition function takes the form

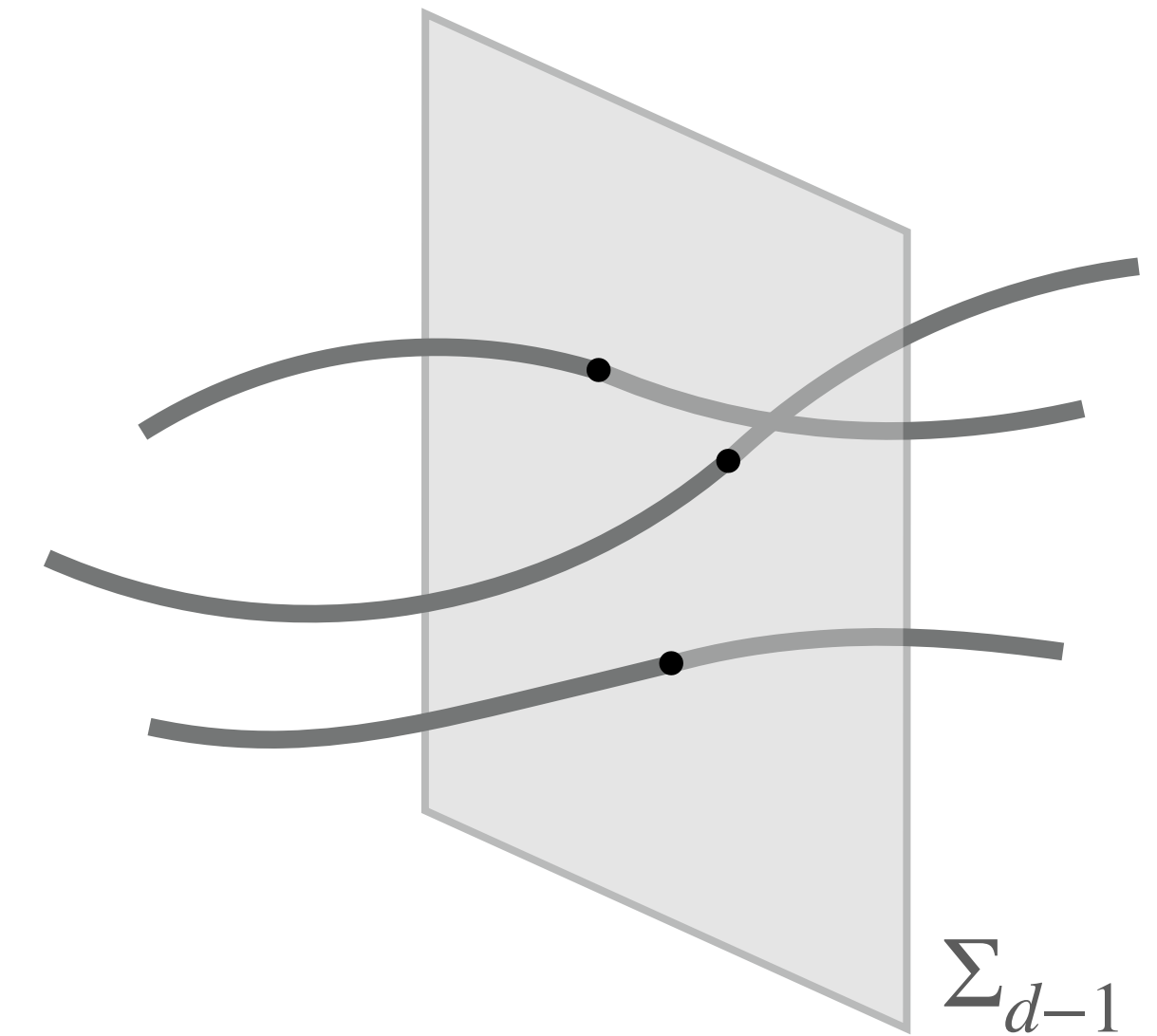
$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \mathcal{D}\phi_i \exp \int d^d x \left(\frac{1}{2} \chi \mu_i \mu^i - \frac{1}{4\tilde{\chi}} \xi_{ij} \xi^{ij} + \dots \right)$$

$$\implies \quad J^{ti} = n^i = \chi \mu^i, \qquad J^{ij} = -\frac{1}{\tilde{\chi}} \xi^{ij}$$

- The configuration equations imply

$$\partial_i J^{ti} = 0 \implies \partial_i \partial^i \phi = 0$$

$$\partial_t J^{ti} + \partial_k J^{ki} = 0 \implies \partial_k \partial^k \phi^i - \partial^i \partial_k \phi^k = 0$$





APPROXIMATE SYMMETRIES IN EQB.

- When symmetries are approximate, the thermal partition function can in addition depend on the background phase fields: Φ for 0-form symmetries and Φ_μ for one-form symmetries.

APPROXIMATE 0-FORM HYDROSTATICS

- ▶ In the spontaneously unbroken phase, explicit breaking of 0-form symmetry does not do anything interesting. This is because there are no particle creation/annihilation effects in equilibrium.
- ▶ In the superfluid phase, we do see new physics in terms of mass term for the Goldstone

$$\mathcal{L}[A, \Phi] = \int \mathcal{D}\phi \exp \int d^d x \left(\frac{1}{2} \chi \mu^2 - \frac{1}{2} f_s \xi_i \xi^i - \frac{1}{2} m^2 \psi^2 + \dots \right) \quad \psi = \ell(\phi - \Phi)$$

$$\implies \quad J^t = n = \chi \mu \quad J^i = -f_s \xi^i$$

$$L = m^2 \psi$$

- ▶ Classical configuration equation for ϕ impose the conservation equations in equilibrium

$$\partial_t J^t + \partial_i J^i = -\ell L \implies \partial_i \partial^i \phi = k_{0\phi}^2 \phi$$

$$\frac{1}{k_{0\phi}} = \sqrt{\frac{f_s}{\ell^2 m^2}}$$

“Pinning length”

APPROXIMATE 1-FORM HYDROSTATICS

- The partition function can also depend on Φ_μ through the **defect chemical potential**

$$\mu_\ell = -\ell (\varphi - \Phi_t)$$

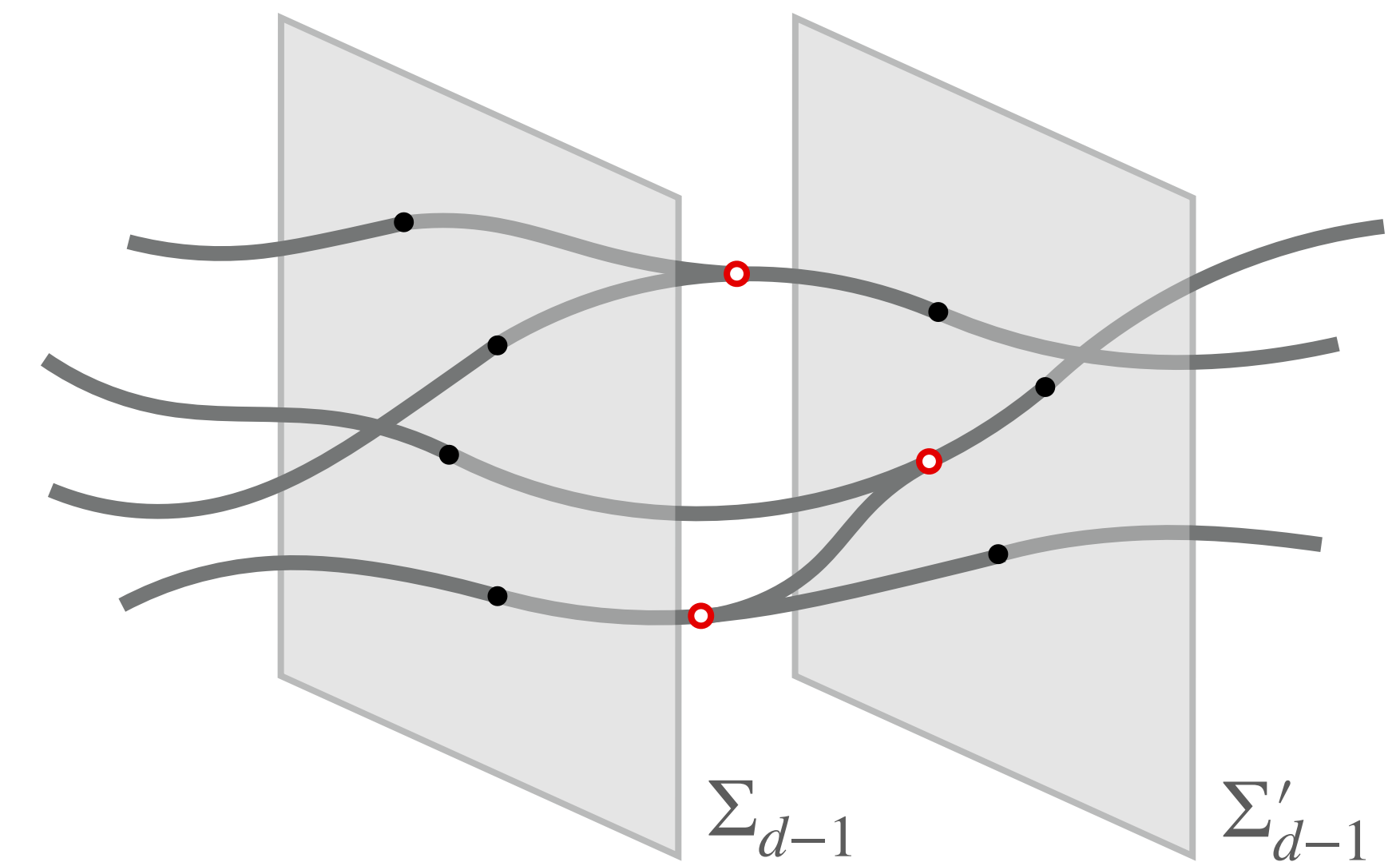
$$\Phi_t \rightarrow \Phi_t + \partial_t \Lambda_\ell$$

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int d^d x \left(\frac{1}{2} \chi \mu_i \mu^i + \frac{1}{2} \chi_\ell \mu_\ell^2 + \dots \right)$$

$$\implies J^{ti} = n^i = \chi \mu^i \quad J^{ij} = 0$$

$$L^t = n_\ell = \chi_\ell \mu_\ell \quad L^i = 0$$



e.g. electrostatics **free charges** or magnetostatics with **monopoles**.

- Classical configuration equation for φ imposes the **Gauss constraint**

$$\partial_i J^{ti} = \ell L^t \implies \partial_i \partial^i \varphi = k_0^2 \varphi$$

$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2 \chi_\ell}}$$

“Debye length”

APPROXIMATE 1-FORM SUPERFLUID HYDROSTATICS

- In the presence of explicit symmetry breaking, the 1-form superfluid can exist in two phases depending on the **0-form defect symmetry** being spontaneously broken or not.
- In the **relaxed/Coulomb phase**, the 0-form defect symmetry is **spontaneously unbroken** and we can only construct the “defect chemical potential”

$$\mu_\ell = -\ell (\varphi - \Phi_t)$$

$$\Phi_i \rightarrow \Phi_i + \partial_i \Lambda_\ell$$

- In the **pinned/Higgs phase**, the 0-form defect symmetry is **spontaneously broken**

$$\phi_\ell \rightarrow \phi_\ell - \Lambda_\ell$$

$$\nabla_\mu J^{\mu\nu} = \ell L^\nu$$

This allows us to also construct the **phase misalignment vector**

$$\nabla_\mu L^\mu = 0$$

$$\psi_i = \ell (\phi_i - \Phi_i - \partial_i \phi_\ell)$$

In the language of Higgs mechanism, the 1-form phase ϕ_i eats the 0-form phase ϕ_ℓ to become massive.

APPROXIMATE 1-FORM SUPERFLUID HYDROSTATICS

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \mathcal{D}\phi_i \exp \int d^3x \left(\frac{1}{2} \chi \mu_i \mu^i - \frac{1}{4\tilde{\chi}} \xi_{ij} \xi^{ij} + \frac{1}{2} \chi_\ell \mu_\ell^2 - \frac{m^2}{2} \psi_i \psi^i + \dots \right)$$

$$\implies J^{ti} = n^i = \chi \mu^i, \quad J^{ij} = -\frac{1}{\tilde{\chi}} \xi^{ij}$$

$$L^t = n_\ell = \chi_\ell \mu_\ell \quad L^i = m^2 \psi^i$$

e.g. electromagnetism with **free charges** and/or **massive gauge fields**.

- Classical configuration equation for φ, ϕ_i impose the conservation equations in equilibrium

$$\partial_i J^{ti} = \ell L^t \implies \partial_i \partial^i \varphi = k_0^2 \varphi$$

$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2 \chi_\ell}}$$

$$\frac{1}{k_{0\phi}} = \sqrt{\frac{1}{\ell^2 m^2 \tilde{\chi}}}$$

$$\partial_t J^{ti} + \partial_k J^{ki} = -\ell L^i \implies \partial_k \partial^k \phi^i - \partial^i \partial_k \phi^k = k_{0\phi}^2 \phi^i$$

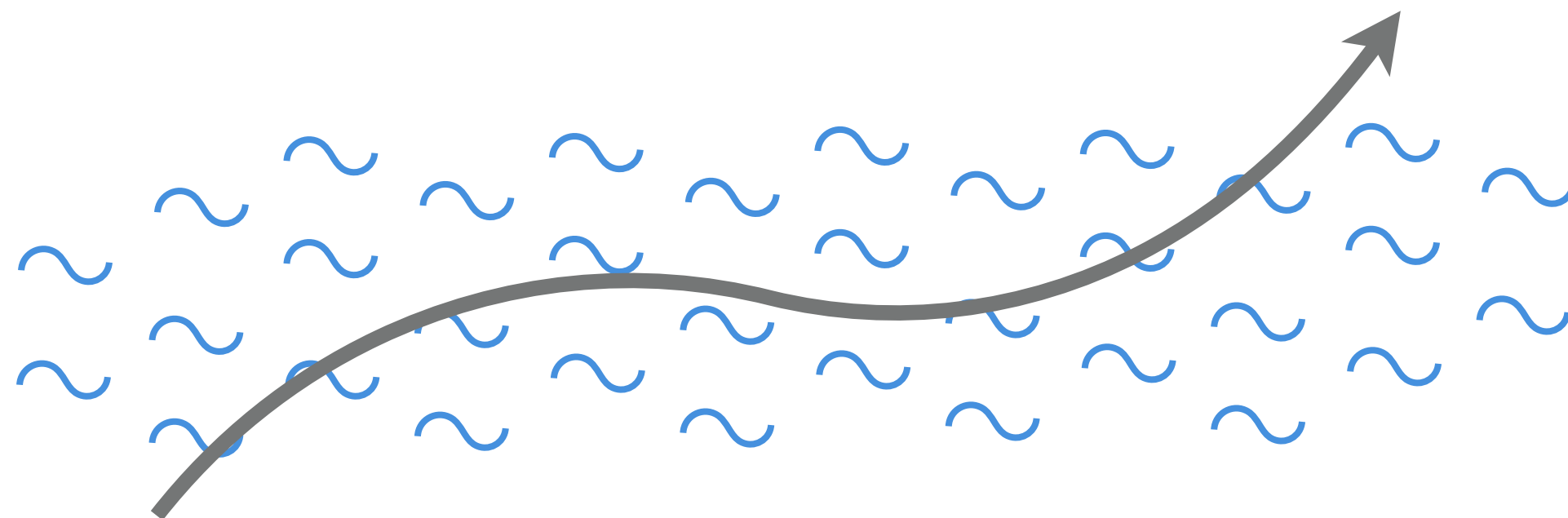
“Debye length”

“London depth”



HYDRODYNAMICS

- **Hydrodynamics** is a framework to capture perturbative departures of a many-body system from thermal equilibrium.
- The relevant hydrodynamic degrees of freedom are a set of **symmetry parameters** corresponding to each global symmetry (conserved charge) of the system.
- Additionally, we need to add massless **Goldstone fields** for each spontaneously broken global symmetry.



0-FORM HYDRODYNAMICS

- ▶ The hydrodynamic description is based on the **conservation laws**

$$\nabla_{\mu} J^{\mu} = -\ell L, \quad \nabla_{\mu} T^{\mu\nu} = F^{\nu\rho} J_{\rho} + \Xi^{\nu} L$$

we have allowed for violation of the 0-form symmetry.

- ▶ The **hydrodynamic fields** are $\Lambda_{\beta}, \beta^{\mu}$ transforming as

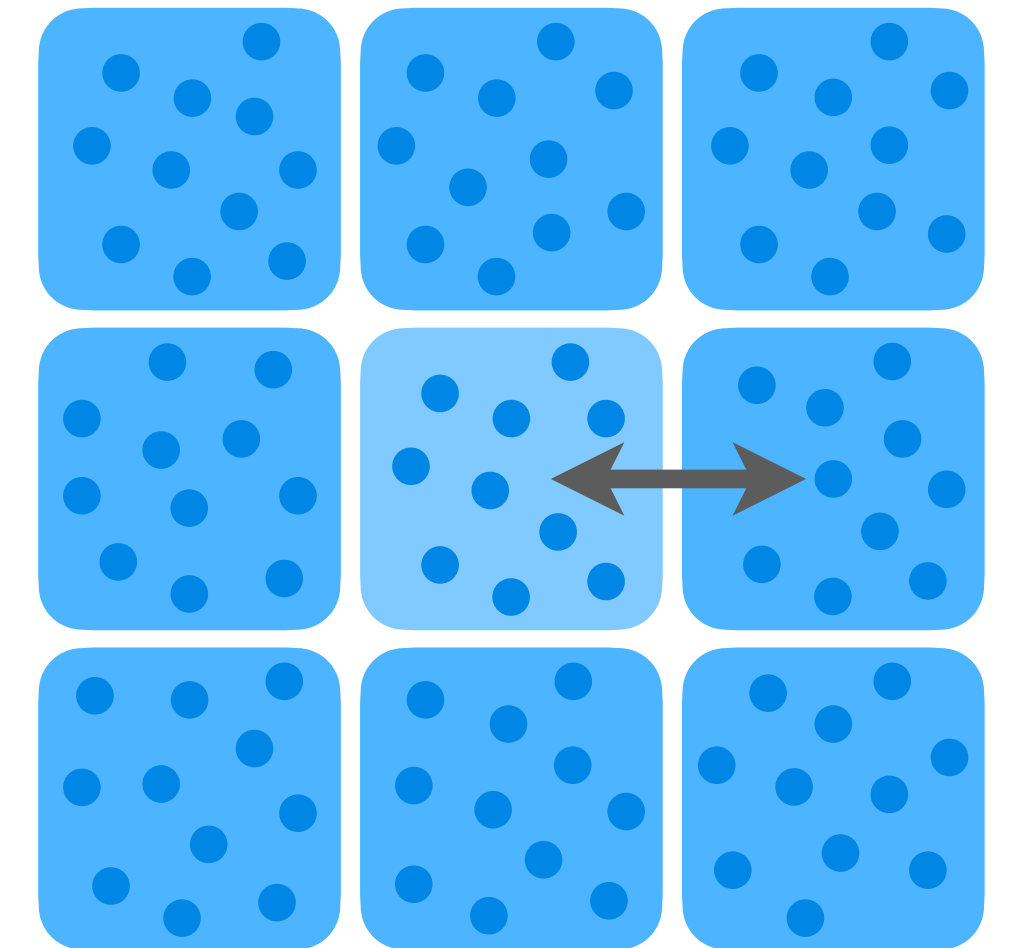
$$\delta\Lambda_{\beta} = \mathfrak{L}_{\chi}\Lambda_{\beta} - \mathfrak{L}_{\beta}\Lambda, \quad \delta\beta^{\mu} = \mathfrak{L}_{\chi}\beta^{\mu}$$

- ▶ These can be used to define gauge-invariant hydrodynamic fields μ, T, u^{μ} as

$$\frac{\mu}{T} = \Lambda_{\beta} + \beta^{\mu} A_{\mu}, \quad \frac{u^{\mu}}{T} = \beta^{\mu}$$

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$$

$$\Xi_{\mu} = \partial_{\mu}\Phi + A_{\mu}$$



0-FORM HYDRODYNAMICS

- Hydrodynamics is characterised by its **constitutive relations**

$$J^\mu, L, T^{\mu\nu} \quad \text{in terms of} \quad \mu, T, u^\mu, A_\mu, \Phi, g_{\mu\nu}$$

- Imposing the **second law of thermodynamics**, we find the constitutive relations

$$J^\mu = n u^\mu - \sigma P^{\mu\nu} \left(T \partial_\nu \frac{\mu}{T} + u^\lambda F_{\lambda\nu} \right)$$

$$L = -\ell \sigma_\ell \left(u^\mu \Xi_\mu - \mu \right)$$

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} - 2\eta P^{\mu\rho} P^{\nu\sigma} \nabla_{\langle\rho} u_{\sigma\rangle} - \zeta P^{\mu\nu} \nabla_\lambda u^\lambda$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$\delta p = s \delta T + n \delta \mu$$

$$\epsilon = Ts + \mu n - p$$

$$\sigma, \sigma_\ell, \eta, \zeta \geq 0$$

- The coefficient σ_ℓ results in **charge relaxation**

$$\nabla_\mu J^\mu = -\ell L \quad \implies \quad u^\mu \partial_\mu n + \dots = -\ell^2 \sigma_\ell \mu(n) + \dots$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu = 0$ state.

In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the **fluid sound** and **shear modes**

$$u_{\parallel}, T : \quad \omega = \pm v_s k - \frac{i}{2} D_{\pi}^{\parallel} k^2 + \dots$$

$$u_{\perp} : \quad \omega = -i D_{\pi}^{\perp} k^2 + \dots$$

$$v_s^2 = \frac{\partial p}{\partial \epsilon}$$

$$D_{\pi}^{\parallel} = \frac{\zeta + 2 \frac{d-1}{d} \eta}{\epsilon + p}, \quad D_{\pi}^{\perp} = \frac{\eta}{\epsilon + p}$$

- The charge fluctuations give rise to a **diffusive mode**

$$\mu : \quad \omega = -i D_n k^2 - i \Gamma$$

$$D_n = \frac{\sigma}{\chi}, \quad \Gamma = \frac{\ell^2 \sigma_{\ell}}{\chi}, \quad \chi = \frac{\partial n}{\partial \mu}$$

Charge fluctuations are **damped** due to explicit symmetry breaking.

1-FORM HYDRODYNAMICS

- The hydrodynamic description is based on the **conservation laws**

$$\nabla_{\mu} J^{\mu\nu} = \ell L^{\nu}$$

$$\nabla_{\mu} L^{\mu} = 0$$

$$\nabla_{\mu} T^{\mu\nu} = \frac{1}{2} F^{\nu\rho\sigma} J_{\rho\sigma} + \ell \Xi^{\nu\rho} L_{\rho}$$

$$F_{\mu\nu\rho} = 3\partial_{[\mu} A_{\nu\rho]}$$

$$\Xi_{\mu\nu} = 2\partial_{[\mu} \Phi_{\nu]} + A_{\mu\nu}$$

- The hydrodynamic fields are Λ_{μ}^{β} , Λ_{ℓ}^{β} , β^{μ} , together with φ , transforming as

$$\delta\Lambda_{\mu}^{\beta} = \mathfrak{L}_{\chi}\Lambda_{\mu}^{\beta} - \mathfrak{L}_{\beta}\Lambda_{\mu}$$

$$\delta\beta^{\mu} = \mathfrak{L}_{\chi}\beta^{\mu}$$

$$\delta\varphi = \mathfrak{L}_{\chi}\varphi - \beta^{\mu}\Lambda_{\mu}$$

$$\delta\Lambda_{\ell}^{\beta} = \mathfrak{L}_{\chi}\Lambda_{\ell}^{\beta} - \mathfrak{L}_{\beta}\Lambda_{\ell}$$

- These can be used to define the gauge-invariant hydrodynamic fields

$$\frac{\mu_{\mu}}{T} = \Lambda_{\mu}^{\beta} + \beta^{\lambda} A_{\lambda\mu} - \partial_{\mu}\varphi, \quad \frac{\mu_{\ell}}{T} = -\ell \left(\varphi - \beta^{\mu}\Phi_{\mu} - \Lambda_{\ell}^{\beta} \right), \quad \frac{u^{\mu}}{T} = \beta^{\mu}$$

JOSEPHSON EQUATION FOR TEMPORAL GOLDSTONE

- ▶ The dynamics of φ is governed by a **Josephson equation** of the form

$$\mathfrak{L}_\beta \varphi = \beta^\mu \Lambda_\mu^\beta + \dots$$

- ▶ We can absorb possible corrections to this equation by redefining Λ_μ^β . This implies

$$u^\mu \mu_\mu = u^\mu \Lambda_\mu^\beta - u^\mu \partial_\mu \varphi = 0$$

Therefore, the 1-form chemical potential is **purely spatial**.

GAUGE REDUNDANCY

- There is a **gauge redundancy** in the description that can be obtained by setting

$$\Lambda_\mu = -\partial_\mu \lambda, \quad \Lambda_\ell = -\lambda$$

which leaves the background fields $A_{\mu\nu}$, Φ_μ invariant.

- The dynamical fields transform as

$$\delta_\lambda \Lambda_\mu^\beta = \partial_\mu \mathfrak{L}_\beta \lambda$$

$$\delta_\lambda \beta^\mu = 0$$

$$\delta_\lambda \varphi = \mathfrak{L}_\beta \lambda$$

$$\delta_\lambda \Lambda_\ell^\beta = \mathfrak{L}_\beta \lambda$$

The physical hydrodynamic fields μ_μ , μ_ℓ , T , u^μ are invariant under these gauge transformations.

1-FORM HYDRODYNAMICS

- The **constitutive relations** are a straight-forward generalisation of the 0-form case

$$J^{\mu\nu} = 2u^{[\mu}n^{\nu]} - \sigma P^{\mu\rho}P^{\nu\sigma} \left(2T\partial_{[\rho} \frac{\mu_{\sigma]} }{T} + u^\lambda F_{\lambda\rho\sigma} \right)$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$L^\mu = n_\ell u^\mu - \sigma_\ell P^{\mu\nu} \left(T\partial_\nu \frac{\mu_\ell}{T} + \ell u^\lambda \Xi_{\lambda\nu} - \ell \mu_\nu \right)$$

$$\delta p = s\delta T + n\delta\mu + n_\ell \delta\mu_\ell$$

$$\epsilon = Ts + \mu n + \mu_\ell n_\ell - p$$

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu} - \chi \mu^\mu \mu^\nu - 2\eta P^{\mu\rho}P^{\nu\sigma} \nabla_{\langle\rho} u_{\sigma\rangle} - \zeta P^{\mu\nu} \nabla_\lambda u^\lambda$$

$$\sigma, \sigma_\ell, \eta, \zeta \geq 0$$

- Note that σ_ℓ now behaves like a **conductivity** for the defect flux, as well as results in **string charge relaxation**

$$\begin{aligned} \nabla_\mu J^{\mu\nu} = -\ell L^\nu \quad \implies \quad & u^\mu \partial_\mu n^\nu + \dots = -\ell^2 \sigma_\ell \mu^\nu(n) + \ell \sigma_\ell P^{\nu\rho} \partial_\rho \mu_\ell + \dots \\ & -\partial_\mu n^\mu u^\nu \quad -\ell n_\ell u^\nu \end{aligned}$$

LINEARISED FLUCTUATIONS

- ▶ Let us assume that we are fluctuating around $\mu_\mu = 0$ state.

In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same **fluid sound** and **shear modes**.

- ▶ The charge fluctuations give rise to two **diffusive modes**

$$\mu_\perp : \quad \omega = -iD_n k^2 - i\Gamma$$

$$\mu_\parallel : \quad \omega = -iD_\ell k^2 - i\Gamma$$

$$D_n = \frac{\sigma}{\chi}, \quad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}, \quad n_\mu = \chi \mu_\mu$$

$$D_\ell = \frac{\sigma_\ell}{\chi \ell}$$

- ▶ The μ_\parallel mode obeys a **damping-attenuation relation**

$$\Gamma = D_\ell k_0^2$$

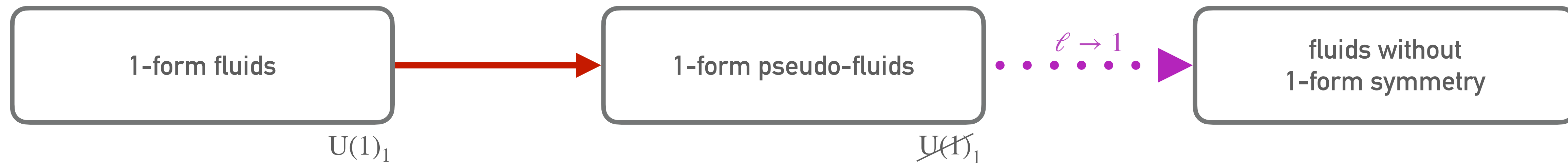
$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2 \chi_\ell}}$$

Such relations are a generic feature of systems where a symmetry is **spontaneously+explicitly** broken.

“Debye length”

DEFECT PROLIFERATION

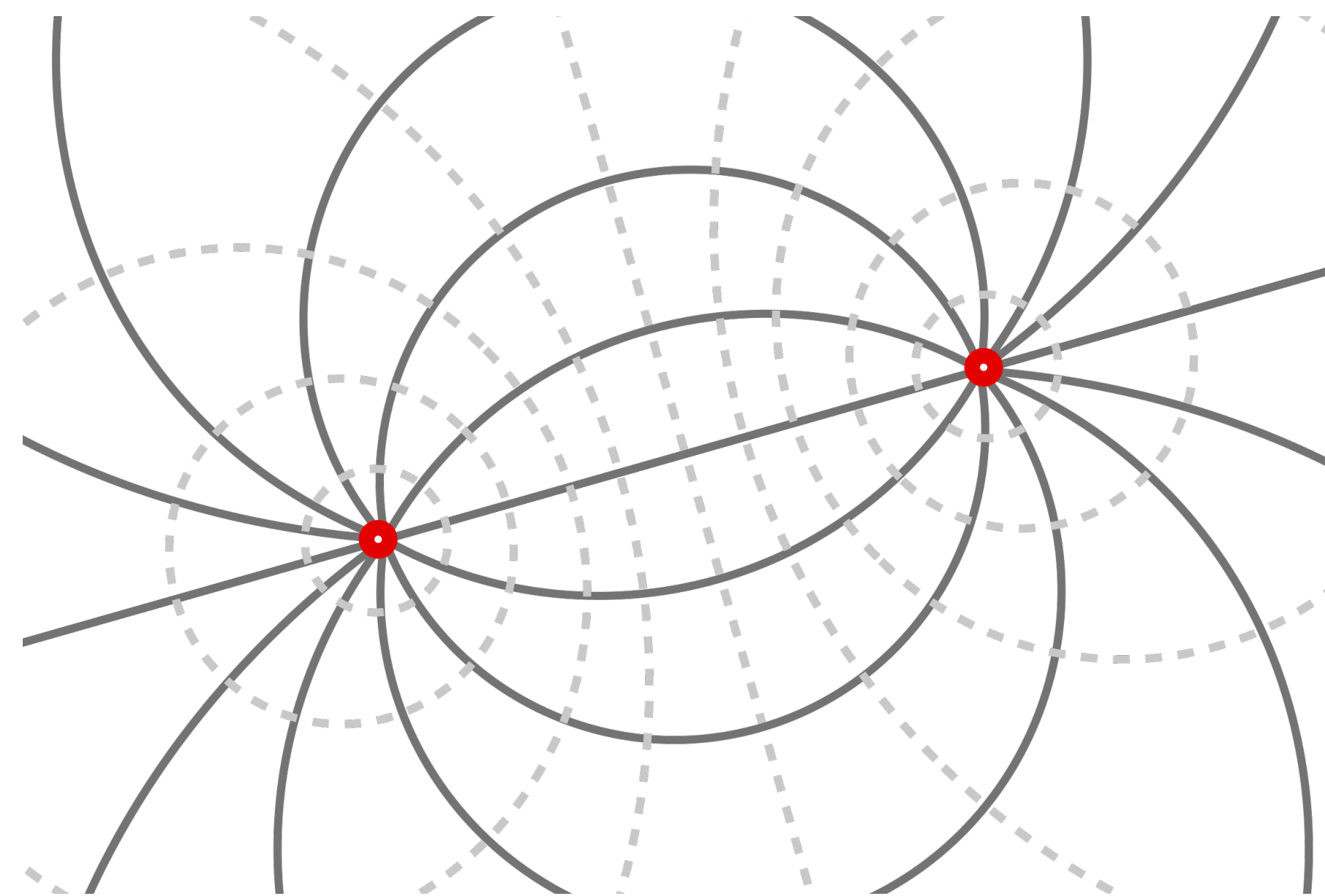
- If we increase the strength of defects, i.e. increase ℓ , the charge fluctuations gap out and we are left with a fluid without 1-form symmetry.



$$u^\mu \partial_\mu n^\nu + \dots = -\Gamma n^\nu + \ell D_\ell P^{\nu\rho} \partial_\rho n_\ell + \dots$$

$$-\partial_\mu n^\mu u^\nu \quad -\ell n_\ell u^\nu$$

- The same discussion also applies to the phase transition of 0-form superfluids to 0-form fluids, mediated by vortices.

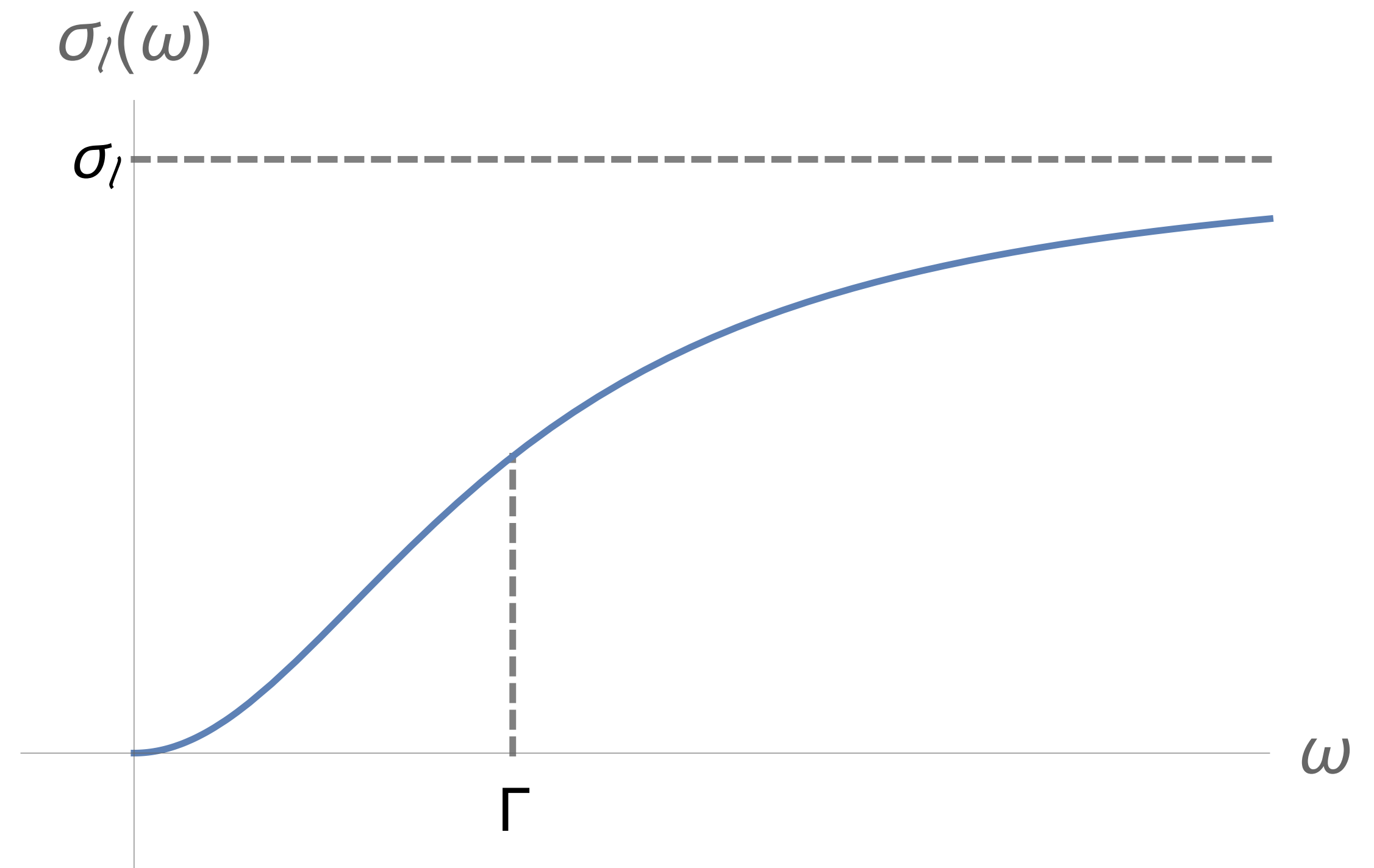


DEFECT PROLIFERATION

► Optical conductivities

$$\sigma(\omega) = \text{Re} \frac{i}{\omega} G_{J_{xy} J_{xy}}^R(\omega) = \sigma$$

$$\sigma_{\ell}(\omega) = \text{Re} \frac{i}{\omega} G_{L^x L^x}^R(\omega) = \sigma_{\ell} \frac{\omega^2 / \Gamma^2}{1 + \omega^2 / \Gamma^2}$$



HIGHER-FORM SUPERFLUIDS

with approximate higher-form
symmetry



LINEARISED 1-FORM SUPERFLUIDS

- ▶ Let us assume that we are fluctuating around $\mu_\mu = \xi_{\mu\nu} = 0$ state.
In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same **fluid sound and shear modes**.
- ▶ The transverse charge and Goldstone fluctuations give rise to the **screened photon mode** in the Coulomb/relaxed phase

$$\mu_\perp, \phi_\perp : (i\omega - D_n k^2 - \Gamma) (i\omega - \tilde{D}_n k^2) + v_\perp^2 k^2 = 0$$

$$v_\perp^2 = \frac{\lambda^2}{\chi \tilde{\chi}}$$

- ▶ In the pinned/Higgs phase, you get additional **pinning** and **screening** effects

$$\mu_\perp, \phi_\perp : (i\omega - D_n k^2 - \Gamma) (i\omega - \tilde{D}_n k^2 - \tilde{\Omega}) + \omega_0^2 + v_\perp^2 k^2 = 0$$

$$\omega_0^2 = v_\perp^2 k_{0\phi}^2$$

$$\tilde{\Omega} = \tilde{D}_n k_{0\phi}^2$$

$$\frac{1}{k_{0\phi}} = \sqrt{\frac{1}{\ell^2 m^2 \tilde{\chi}}}$$

- ▶ Similar results hold for q -form superfluids with vortices or impurities for any q .

“London depth”

LINEARISED 1-FORM SUPERFLUIDS

- ▶ The longitudinal charge mode in the relaxed/Coulomb phase returns the previous **damped diffusive mode**

$$\mu_{\parallel} : \quad i\omega = D_{\ell} k^2 + \Gamma$$

- ▶ For pinned/Higgs phase, this couples to longitudinal Goldstone fluctuation and gives a **second sound mode**

$$\mu_{\parallel}, \phi_{\parallel} : \quad (i\omega - D_{\ell} k^2 - \Gamma) (i\omega - \tilde{D}_{\psi} k^2 - \tilde{\Omega}) + \omega_0^2 + v_{\parallel}^2 k^2 = 0$$

$$\Gamma = D_{\ell} k_0^2$$

$$v_{\parallel}^2 = \frac{\lambda^2 m^2}{\chi_{\ell}}$$

$$\omega_0^2 = v_{\parallel}^2 k_0^2$$

$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2 \chi_{\ell}}}$$

“Debye length”

DEFECT PROLIFERATION

- ▶ We can implement **topological phase transitions** by increasing the strength of ℓ , thereby increasing the strength of explicit symmetry breaking.
- ▶ The product of the phase transition depends on if we are starting from the relaxed/Coulomb phase or pinned/Higgs phase.
- ▶ In the relaxed/Coulomb phase, we arrive at a $(d - 2)$ -**form fluid**. In the context of electromagnetism, this describes **magnetohydrodynamics** with conserved magnetic field lines.

$$\omega = -i \left(\tilde{D}_n + \frac{v_{\perp}^2}{\Gamma} \right) k^2 = -\frac{i}{\tilde{\chi}} \left(\tilde{\sigma} + \frac{\lambda^2}{\sigma_{\ell}} \right) k^2$$

- ▶ In the pinned/Higgs phase, we arrive at a **neutral fluid**. In the context of electromagnetism, this describes the Meissner effect, i.e. expulsion of all electromagnetic fields inside a superconductor.

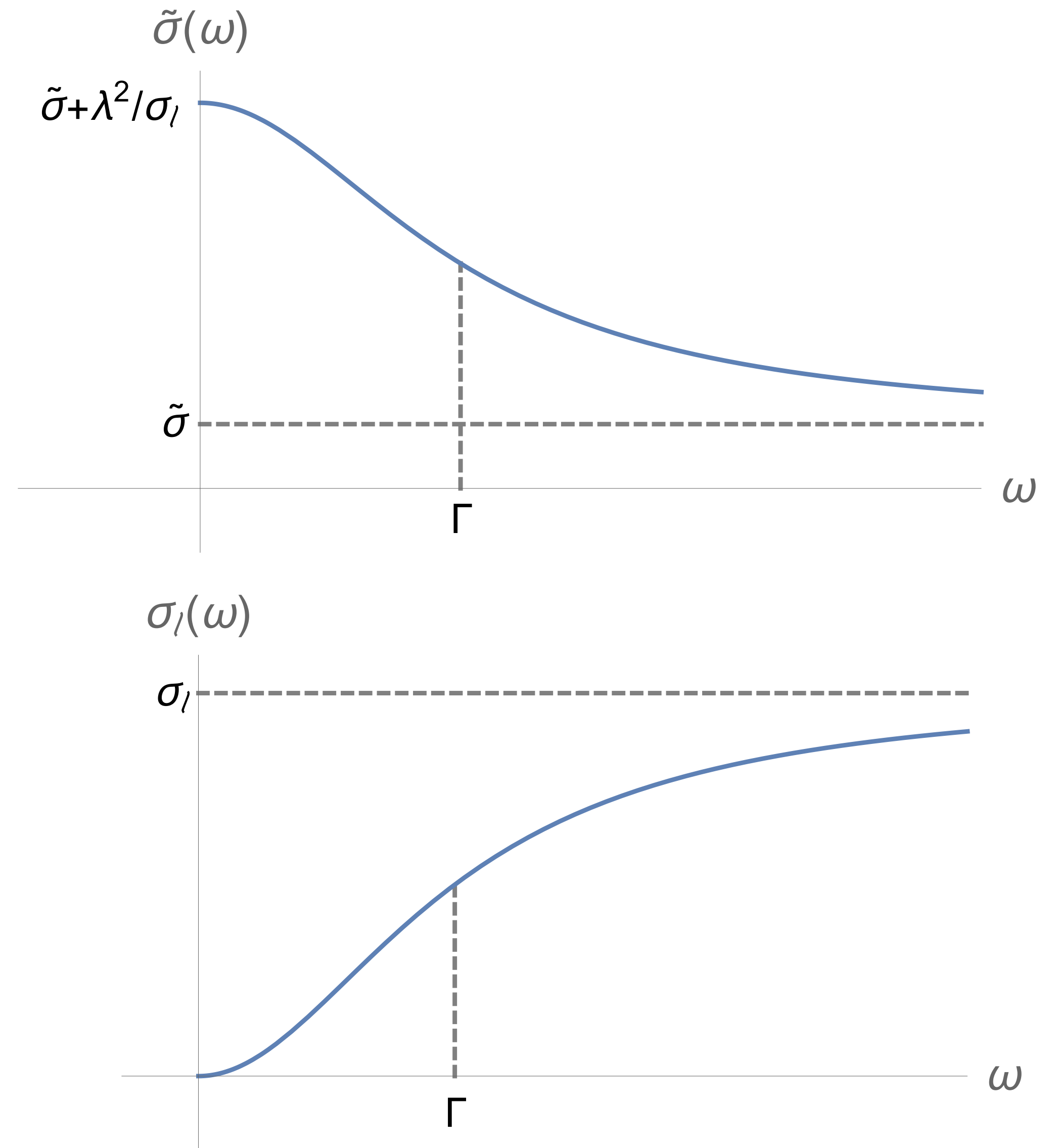
TOPOLOGICAL PHASE TRANSITIONS

- Optical conductivities (Coulomb/relaxed phase)

$$\sigma(\omega) = \text{Re} \frac{i}{\omega} G_{J_{xy} J_{xy}}^R(\omega) = \sigma$$

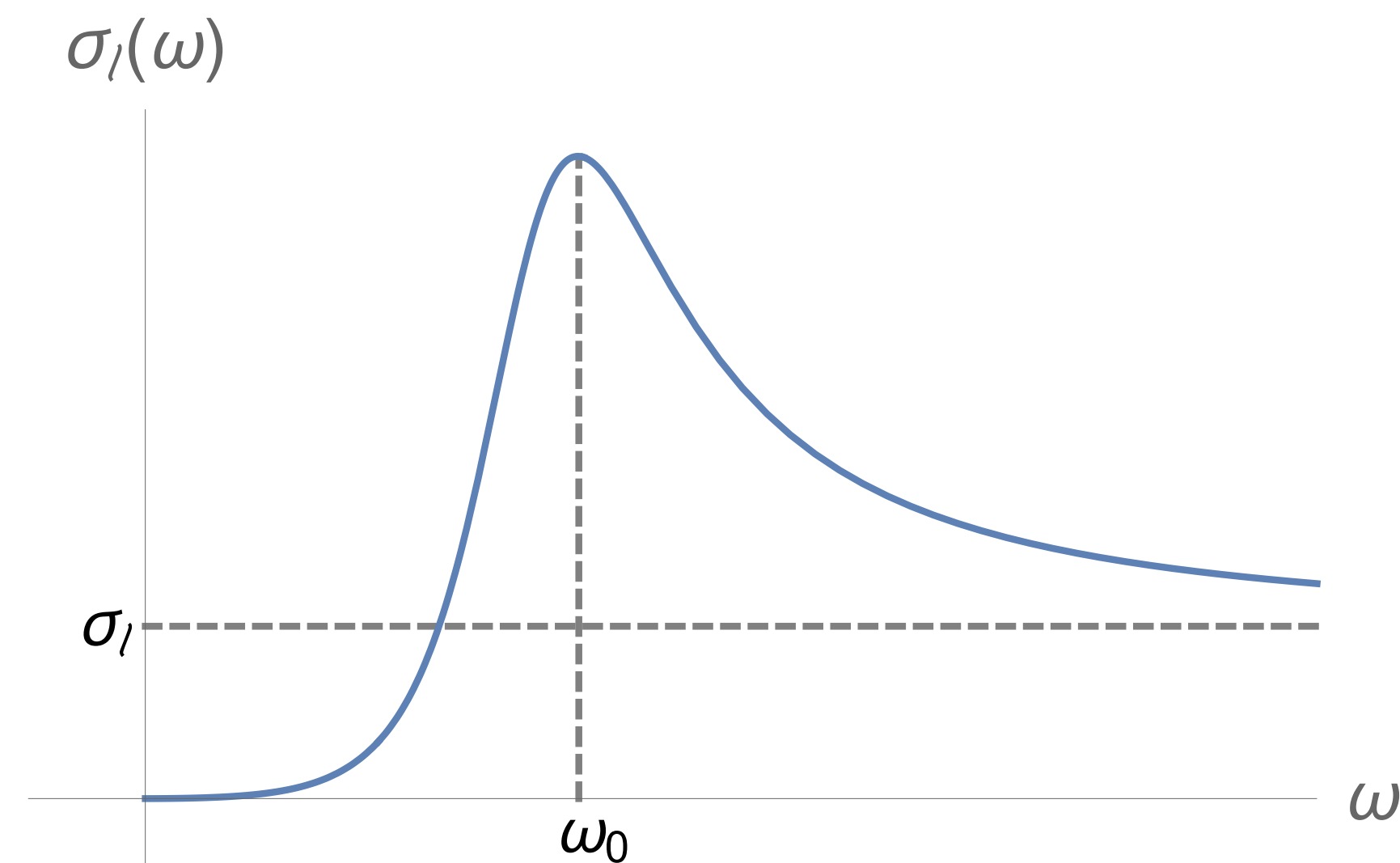
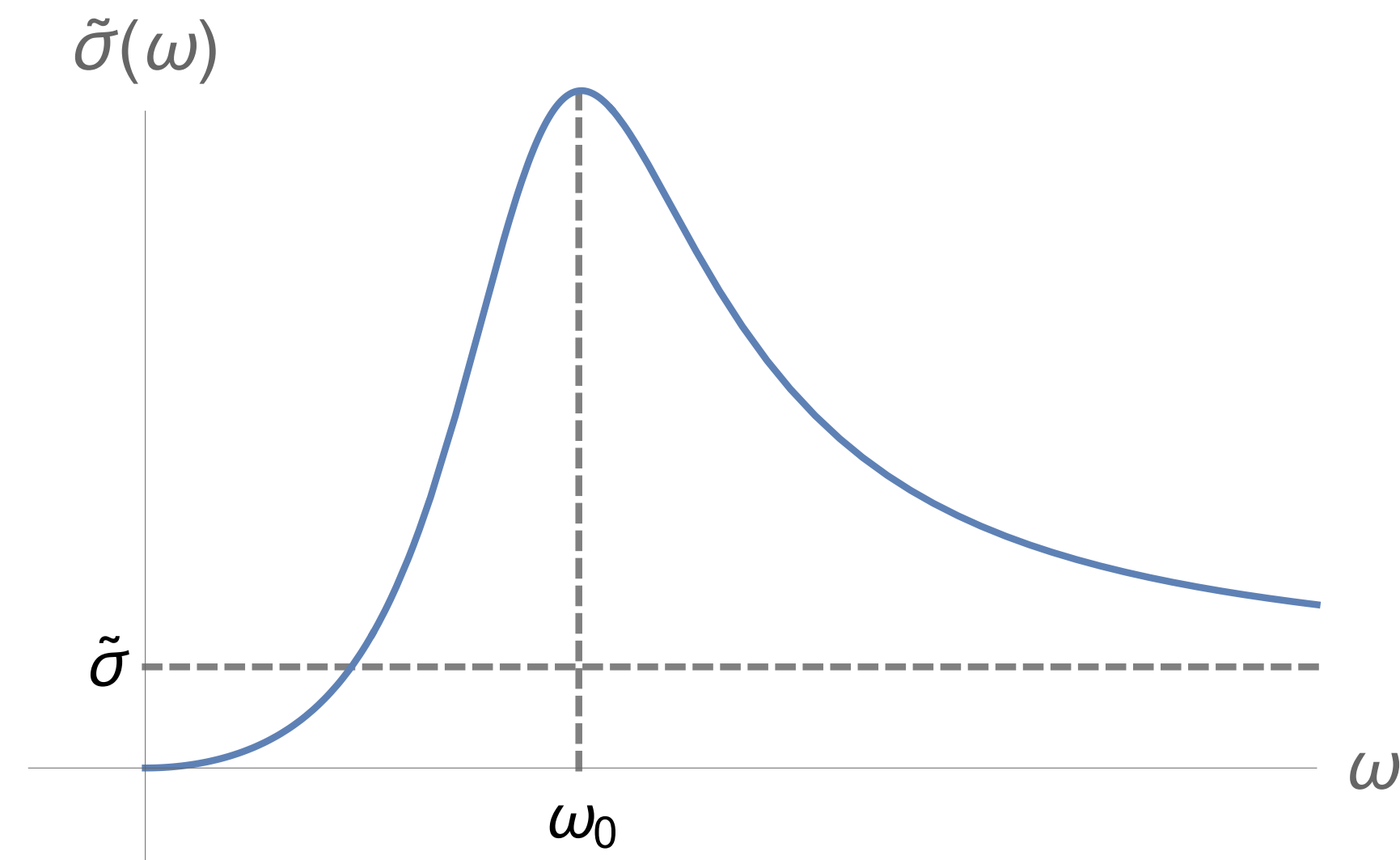
$$\tilde{\sigma}(\omega) = \text{Re} \frac{i}{\omega} G_{\xi_{tx} \xi_{tx}}^R(\omega) = \tilde{\sigma} + \frac{\lambda^2}{\sigma_\ell} \frac{1}{1 + \omega^2/\Gamma^2}$$

$$\sigma_\ell(\omega) = \text{Re} \frac{i}{\omega} G_{L^x L^x}^R(\omega) = \sigma_\ell \frac{\omega^2/\Gamma^2}{1 + \omega^2/\Gamma^2}$$



TOPOLOGICAL PHASE TRANSITIONS

- ▶ Optical correlation functions
(Higgs/pinned phase)





HIGHER-FORM SUPERFLUIDS

- It is possible to also keep the “magnetic” $(d - 2)$ -form symmetry of a 1-form superfluid manifest by coupling the system to a $(d - 1)$ -form gauge field and accounting for the mixed anomaly.
- Explicit breaking of the “magnetic” $(d - 2)$ -form symmetry give rise to vortices in the 1-form superfluid, realising **magnetic monopoles**. Only relaxed/Coulomb phase can admit vortices.
- Extension to q -form superfluids is straight-forward.

OUTLOOK





OUTLOOK

- **Higher-form symmetries** can be used to classify phases of matter with topological order.
- Breaking of continuous higher-form symmetries is associated with **topological defects**, which mediate **topological phase transitions**.
- A hydrodynamic theory with approximate higher-form symmetries provides a model for dynamical phase transitions based on symmetries.
- Further applications include emergent magnetic monopoles in spin ice, plasma phase transitions, melting phase transition in higher-dimensions, superfluid and superconductor phase transitions.

