HIGHER-FORM SYMMETRIES AND TOPOLOGICAL PHASE TRANSITIONS

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[2301.09628] Armas, AJ

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MOTIVATION

- **Symmetries** are powerful guiding principle for developing effective theories for physical systems without a detailed understanding of their microscopic constituents.
- Equilibrium phases of matter can be organised according to their symmetries and whether these are **spontaneously** broken or unbroken in the ground state, commonly known as the **Landau paradigm**.
- Symmetries can even be useful when they are only **approximately** respected by the system.

  The canonical example comes from *chiral perturbation theory*, where pions are seen as pseudo-Goldstones of approximate SU(2) chiral symmetry. Other examples: *pinned crystals, pinned charge density waves, pseudo-superfluids* etc.


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MOTIVATION

- In recent years, the notion of symmetries has been **generalised** to include higher-form symmetries, higher-group symmetries, subsystem symmetries, and non-invertible symmetries.
- These allow for a **generalised Landau paradigm**, that also include exotic phases of matter, such as *topologically ordered states, spin liquids, fractons, topological insulators,* etc.
- The focus of this talk is **continuous higher-form symmetries**, which concerns higher-dimensional charged objects, such as strings and surfaces.
- These describe **topological order** in many-body systems, such as *equipotential planes* in a superfluid, *lattice planes* in a crystal, *magnetic fields* in a plasma, or *electric fields* in a dielectric fluid.

Gaiotto, Kapustin, Seiberg, Willett [1412.5148]


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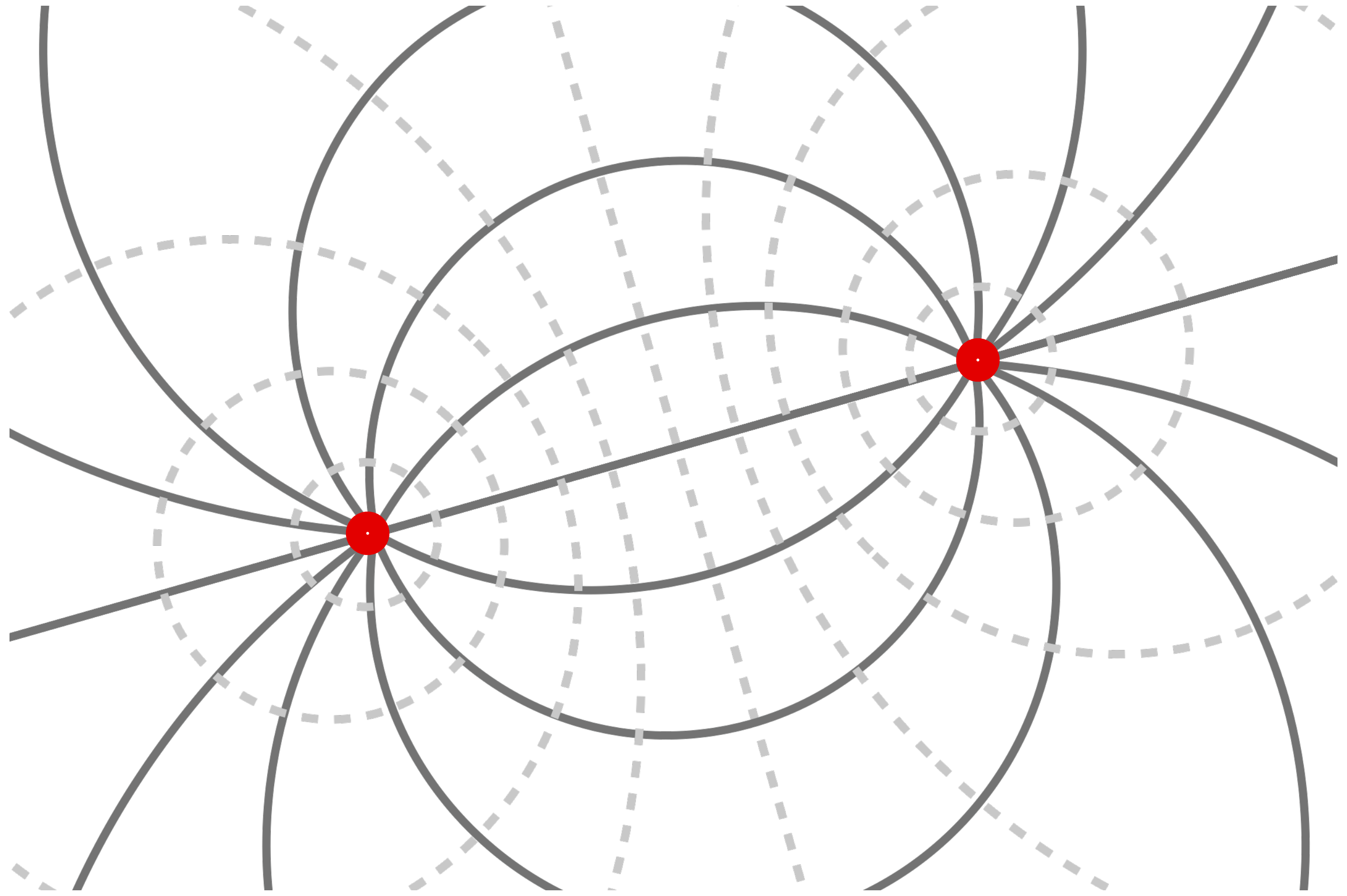
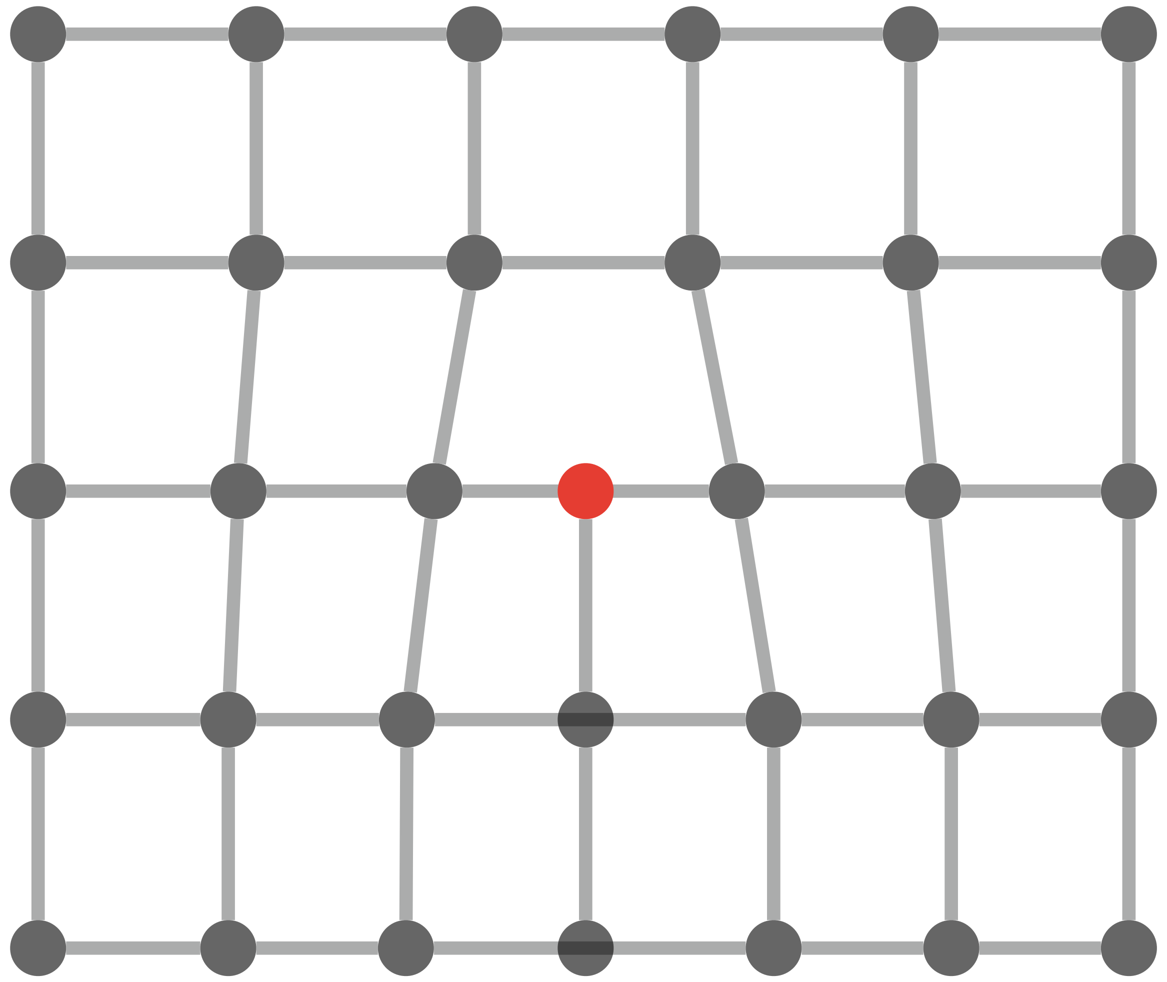
MOTIVATION

- Explicit breaking of higher-form symmetries describes **topological defects**, such as *superfluid vortices, crystal dislocations, magnetic monopoles,* or *free charges*.
- Topological defects mediate **topological phase transitions**,[1] wherein a spontaneously broken symmetry gets restored. Examples include *superfluid phase transition, melting,* and *plasma phase transition*.





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[1] Not to be confused with phase transitions between topologically ordered phases.

HIGHER-FORM SYMMETRIES

and their breaking

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0-FORM SYMMETRIES

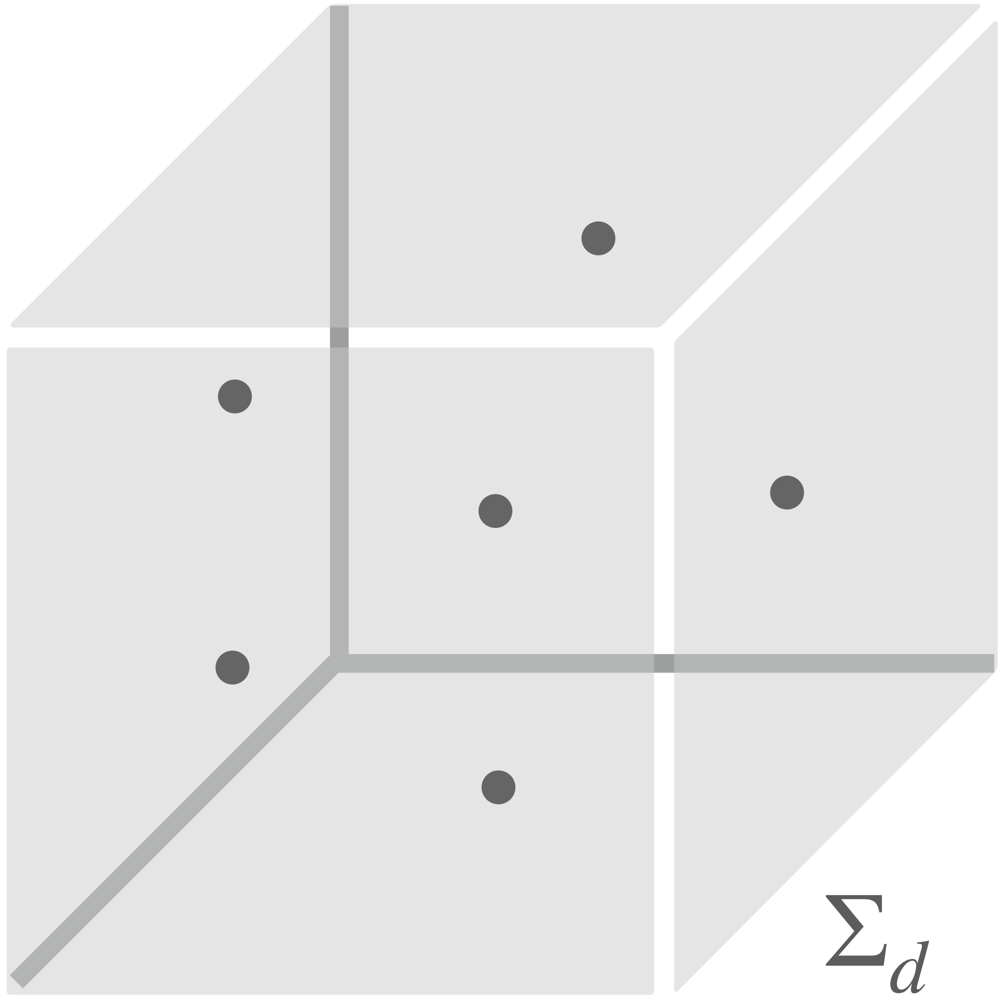
- Continuous 0-form symmetries can be defined by a conservation law

$$\partial_\mu J^\mu = 0 \qquad \Longrightarrow \qquad \partial_t J^t + \partial_i J^i = 0$$

- The total number of charged particles in a volume Σ_d is conserved in time

$$Q[\Sigma_d] = \int \mathrm{d}\Sigma_\mu \, J^\mu = \int \mathrm{d}^d x \, J^t$$

$$\partial_t Q[\Sigma_d] = \int \mathrm{d}^d x \, \partial_t J^t = 0 - \int \mathrm{d}^d x \, \partial_i J^i$$



APPROXIMATE 0-FORM SYMMETRIES

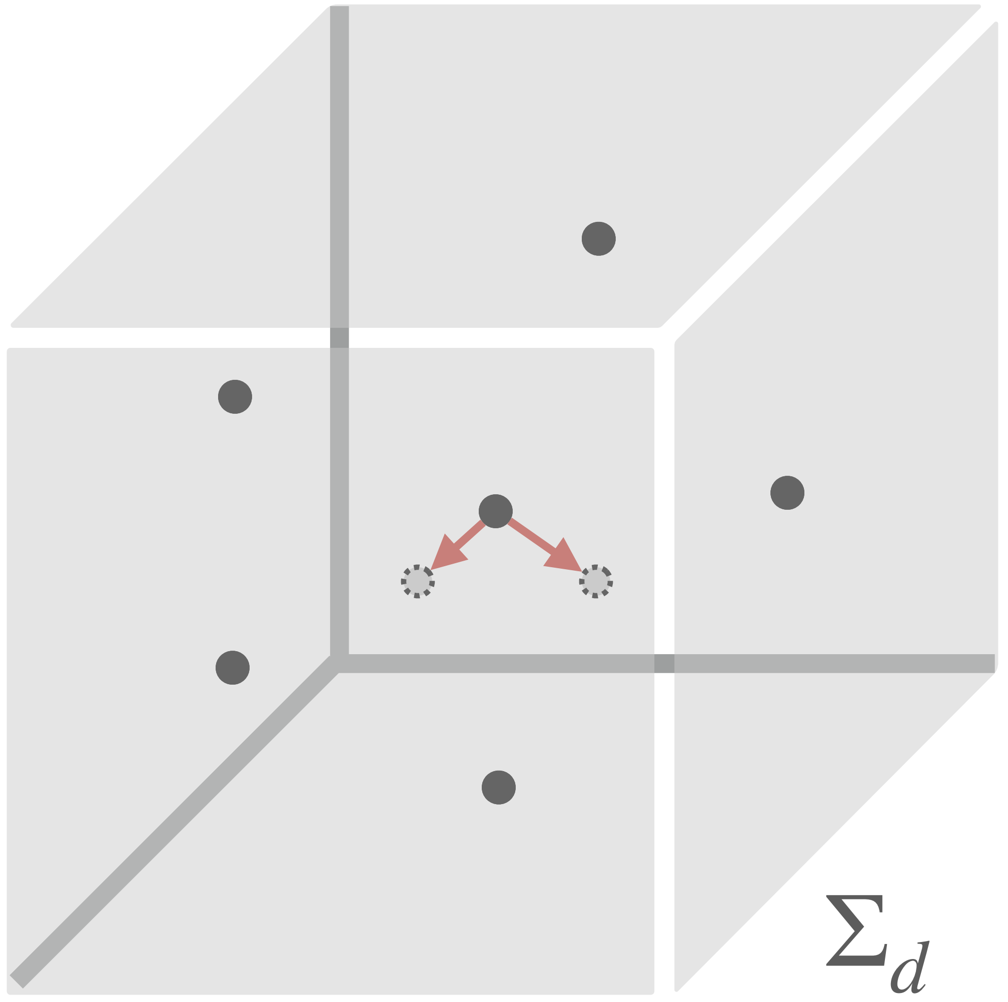
- Approximate 0-form symmetries have weakly violated conservation laws

$$\partial_\mu J^\mu = -\ell L \quad \Longrightarrow \quad \partial_t J^t + \partial_i J^i = -\ell L$$

- The total number of charged particles in a volume Σ_d is only approximately conserved in time

$$Q[\Sigma_d] = \int \mathrm{d}\Sigma_\mu\, J^\mu = \int \mathrm{d}^d x\, J^t$$

$$\partial_t Q[\Sigma_d] = \int \mathrm{d}^d x\, \partial_t J^t = -\ell \int \mathrm{d}^d x\, L - \int \mathrm{d}^d x\, \partial_i J^i$$

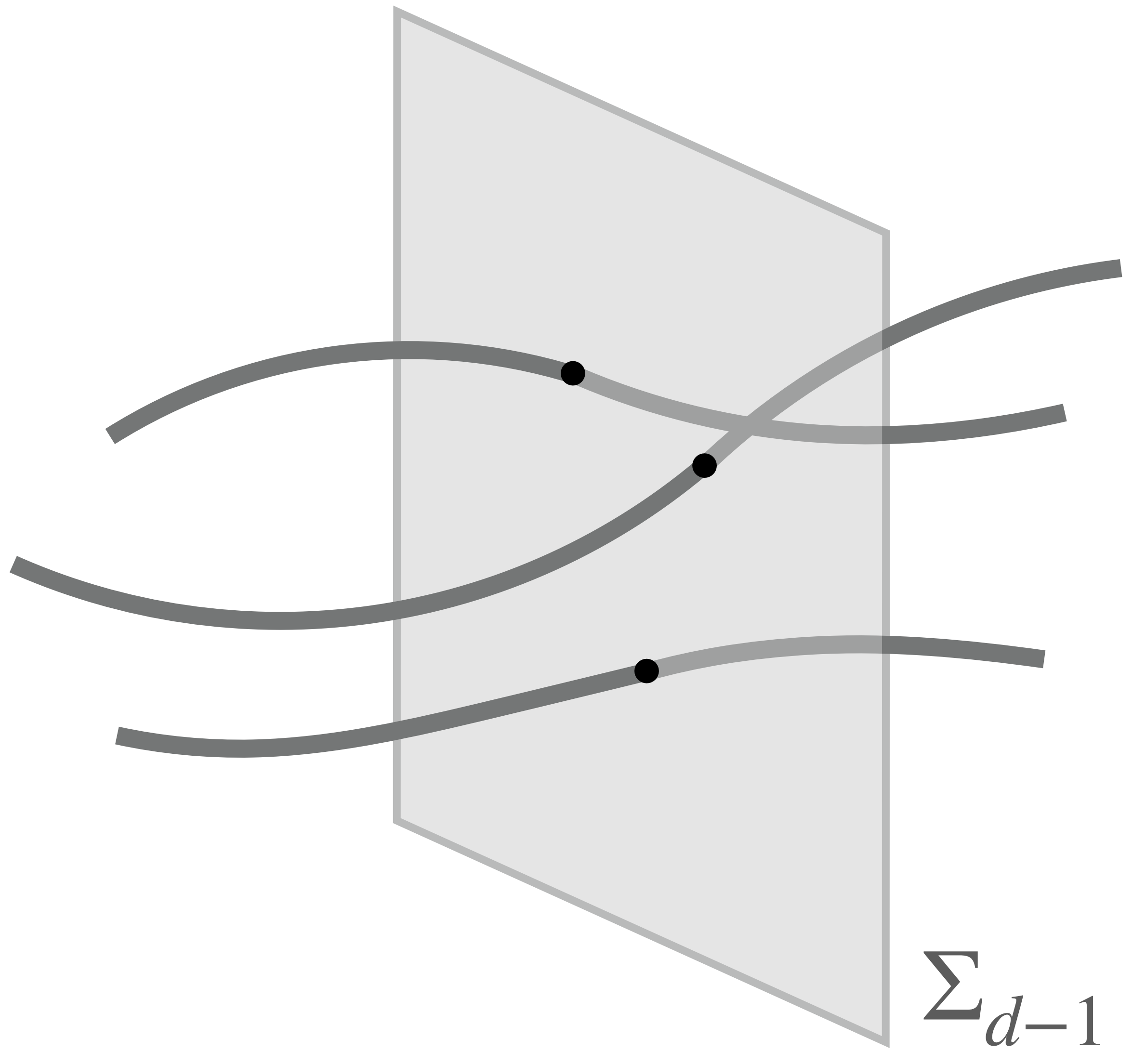


1-FORM SYMMETRIES

- A continuous 1-form symmetry can be defined by the conservation laws

$$\partial_\mu J^{\mu\nu} = 0 \quad \Longrightarrow \quad \begin{aligned} \partial_t J^{ti} + \partial_k J^{ki} &= 0 \\ \partial_i J^{ti} &= 0 \end{aligned}$$

- The objects charged under 1-form symmetries are “strings”.
  The total number of strings passing a cross section Σ_{d-1}
  are conserved under space and time translations



$$Q[\Sigma_{d-1}] = \int \mathrm{d}\Sigma_{\mu\nu}\, J^{\mu\nu} = \int \mathrm{d}^{d-1}x\, J^{tz}$$

$$\partial_t Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_t J^{tz} = 0 \;\; - \int \mathrm{d}^{d-1}x\, \partial_{i_\parallel} J^{i_\parallel z}$$

$$\partial_z Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_z J^{tz} = 0 \;\; - \int \mathrm{d}^{d-1}x\, \partial_{i_\parallel} J^{t i_\parallel}$$

Gaiotto, Kapustin, Seiberg, Willett [1412.5148]

APPROXIMATE 1-FORM SYMMETRIES

- The conservation laws for a 1-form symmetry are

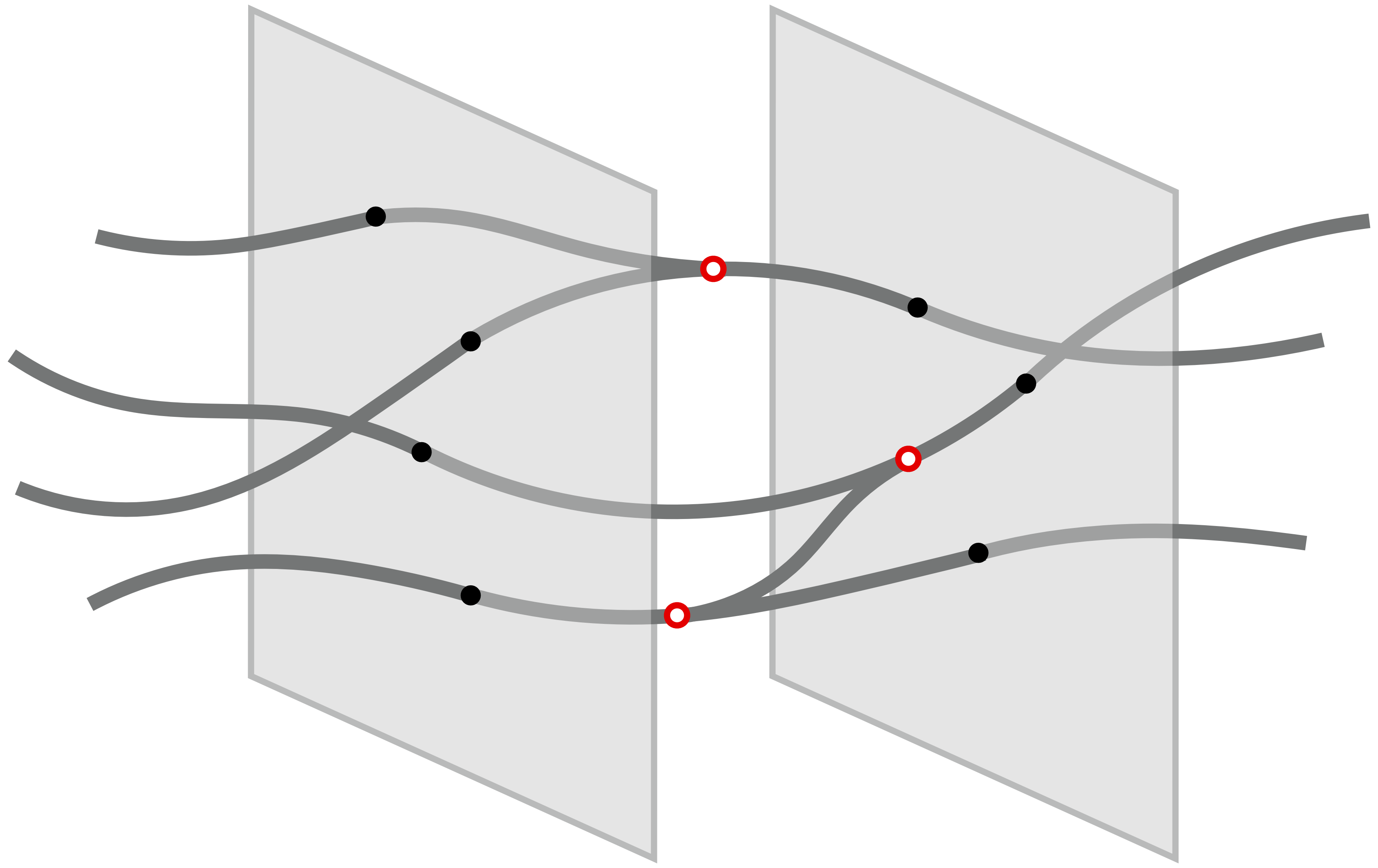
$$\partial_\mu J^{\mu\nu} = -\ell L^\nu \qquad \Longrightarrow \qquad \begin{aligned} \partial_t J^{ti} + \partial_k J^{ki} &= -\ell L^i \\ \partial_i J^{ti} &= \ell L^t \end{aligned}$$

- The total number of strings passing a cross section Σ_{d-1} are only approximately conserved

$$Q[\Sigma_{d-1}] = \int \mathrm{d}\Sigma_{\mu\nu}\, J^{\mu\nu} = \int \mathrm{d}^{d-1}x\, J^{tz}$$

$$\partial_t Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_t J^{tz} = -\ell \int \mathrm{d}^d x\, L^z - \int \mathrm{d}^{d-1}x\, \partial_{i_\|} J^{i_\| z}$$

$$\partial_z Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_z J^{tz} = \ell \int \mathrm{d}^d x\, L^t - \int \mathrm{d}^{d-1}x\, \partial_{i_\|} J^{t i_\|}$$



Armas, AJ [2301.09628]

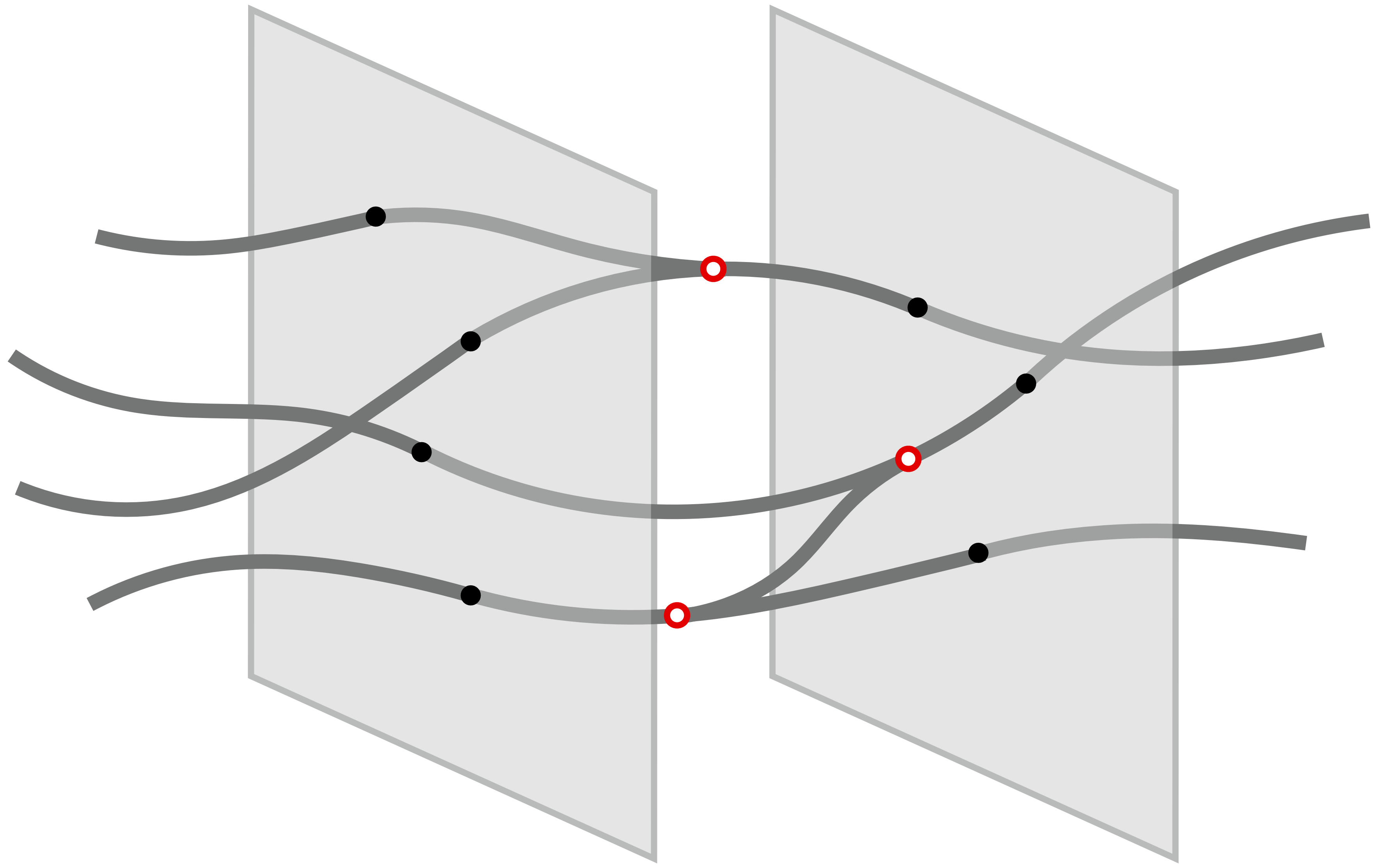
APPROXIMATE 1-FORM SYMMETRIES

- The defects of a 1-form symmetry themselves furnish a 0-form symmetry

$$\partial_\mu L^\mu = 0$$

- The total number of strings passing a cross section Σ_{d-1} are only approximately conserved

$$Q_\ell[\Sigma_d] = \int \mathrm{d}\Sigma_\mu\, L^\mu = \int \mathrm{d}^{d-1}x\, L^t$$



Armas, AJ [2301.09628]

BACKGROUND FIELDS

- We can introduce a 1-form gauge field A_μ and a background phase Φ to probe an approximate 0-form symmetry

$$\delta S[A,\Phi] = \int \mathrm{d}^d x \left(J^\mu \, \delta A_\mu \, + \, \ell L \, \delta \Phi \right)$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \qquad \Phi \to \Phi - \Lambda$$

- Similarly, we can introduce a 2-form gauge field $A_{\mu\nu}$ and a background phase $\Phi_\mu$to probe an approximate 1-form symmetry

$$\delta S[A,\Phi] = \int \mathrm{d}^d x \left(\frac{1}{2} J^{\mu\nu} \, \delta A_{\mu\nu} \, + \, \ell L^\mu \, \delta \Phi_\mu \right)$$

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \qquad \Phi_\mu \to \Phi_\mu - \Lambda_\mu + \partial_\mu \Lambda_\ell$$

Armas, AJ [2301.09628]

EXAMPLE: ELECTROMAGNETISM

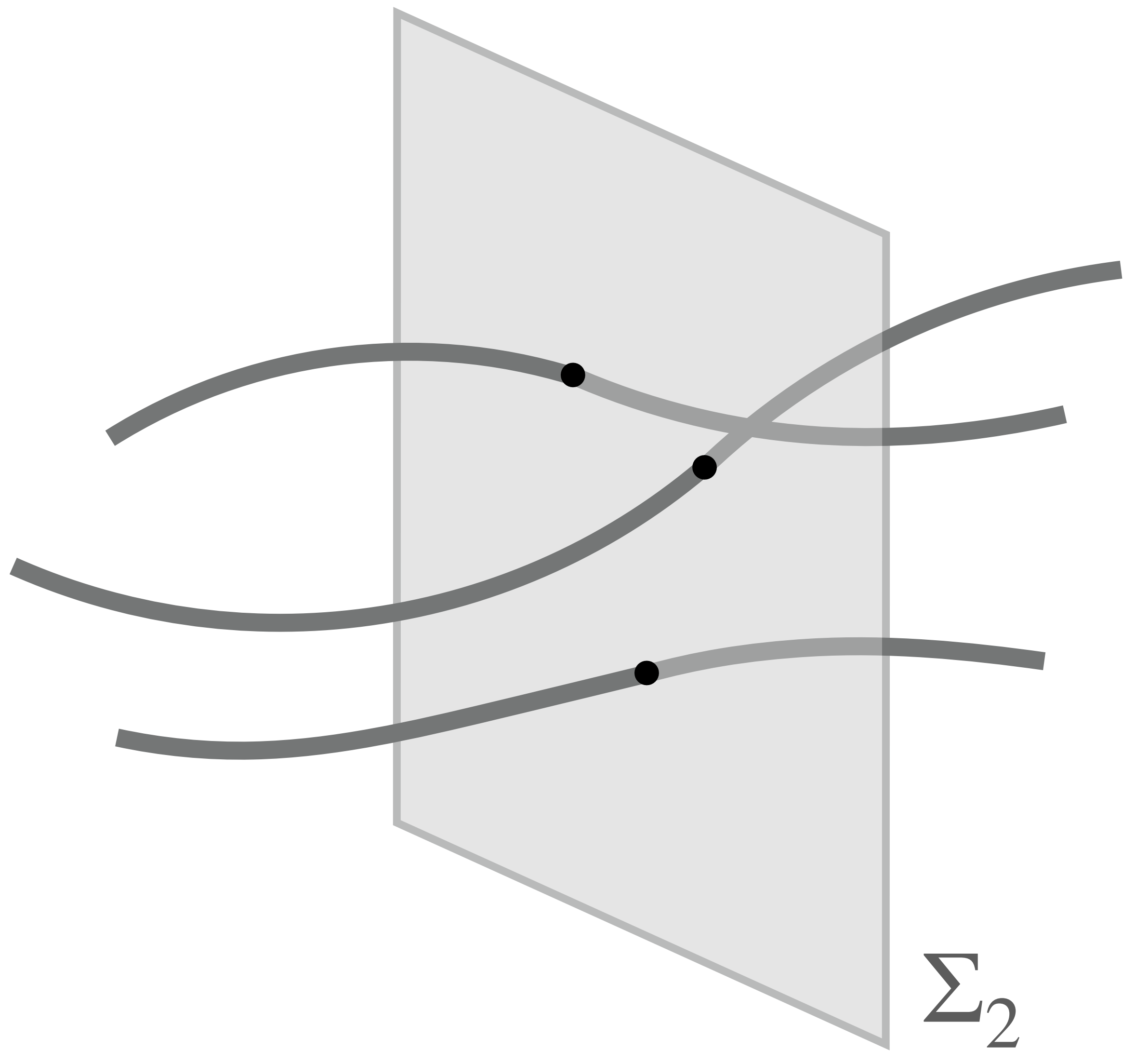
- $d = 3$ electromagnetism in vacuum

$$S = -\int \mathrm{d}^4x \left(\frac{1}{4} \mathscr{F}^{\mu\nu} \mathscr{F}_{\mu\nu} \right) \qquad \Longrightarrow \qquad \begin{aligned} \partial_\mu \mathscr{F}^{\mu\nu} &= 0 \\ \partial_\mu \star \mathscr{F}^{\mu\nu} &= 0 \end{aligned}$$

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu$$

- It has two 1-form symmetries:

$$\begin{aligned} J^{\mu\nu} &= -\mathscr{F}^{\mu\nu} \\ \tilde{J}^{\mu\nu} &= \star \mathscr{F}^{\mu\nu} \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \partial_\mu J^{\mu\nu} &= 0 \\ \partial_\mu \tilde{J}^{\mu\nu} &= 0 \end{aligned}$$



The associated charged objects are electric and magnetic field lines.

- These symmetries also persist in the presence of polarised/dielectric matter

$$J^{\mu\nu} = -\mathscr{F}^{\mu\nu} + \mathscr{M}^{\mu\nu},$$

Gaiotto, Kapustin, Seiberg, Willett [1412.5148]

EXAMPLE: ELECTROMAGNETISM

- In the presence of free electric charges, the electric 1-form symmetry is violated, but the magnetic 1-form symmetry persists.

$$S = -\int \mathrm{d}^4x \left(\frac{1}{4}\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} + (\partial_\mu + i\ell q\mathscr{A}_\mu)\Psi^* (\partial^\mu - i\ell q\mathscr{A}^\mu)\Psi + V(\Psi^*\Psi) \right)$$

$$J^{\mu\nu} = -\mathscr{F}^{\mu\nu}$$

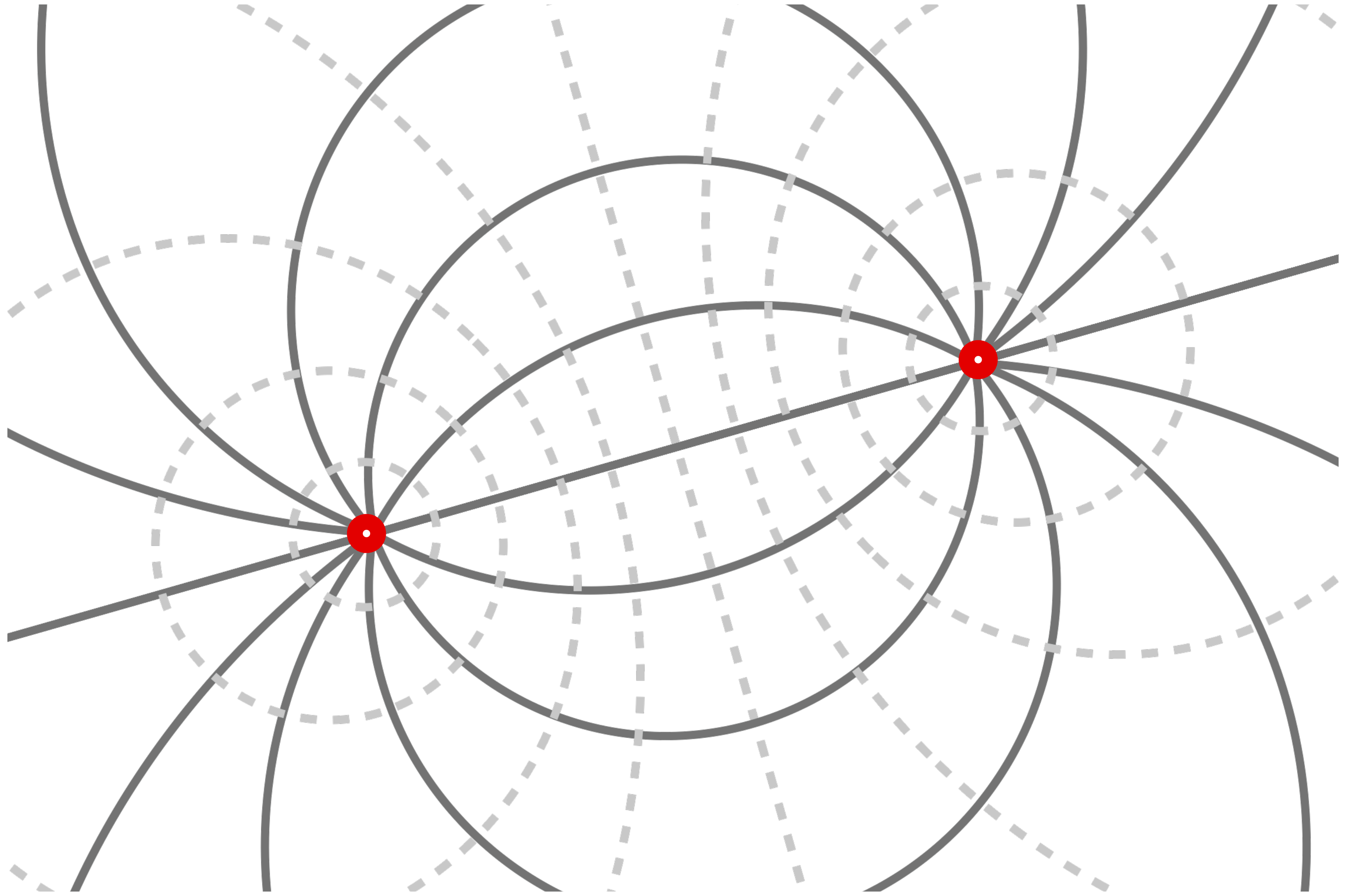
$$\tilde{J}^{\mu\nu} = \star\mathscr{F}^{\mu\nu}$$

$$L^\mu = iq\left(\Psi^*\partial_\mu\Psi - \partial_\mu\Psi^*\Psi\right) + 2\ell q^2\mathscr{A}_\mu\Psi^*\Psi$$

$$\Longrightarrow$$

$$\partial_\mu J^{\mu\nu} = -\ell L^\nu$$

$$\partial_\mu \tilde{J}^{\mu\nu} = 0$$



- Similarly, breaking of the magnetic 1-form symmetry amounts to the introduction of magnetic monopoles.

EXAMPLE: ELECTROMAGNETISM

- We can manifest the electric 1-form symmetry via

$$S[A,\Phi] = -\int \mathrm{d}^4x \left(\frac{1}{4}\xi^{\mu\nu}\xi_{\mu\nu} + (\partial_\mu + i\ell\psi_\mu)\Psi^*(\partial^\mu - i\ell\psi^\mu)\Psi + V(\Psi^*\Psi)\right)$$

$$\xi_{\mu\nu} = \mathscr{F}_{\mu\nu} + A_{\mu\nu}, \qquad \psi_\mu = \ell\left(\mathscr{A}_\mu - \Phi_\mu\right)$$

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \qquad \Phi_\mu \to \Phi_\mu - \Lambda_\mu + \partial_\mu\Lambda_\ell, \qquad \mathscr{A}_\mu \to \mathscr{A}_\mu - \Lambda_\mu, \qquad \Psi \to \mathrm{e}^{-i\ell\Lambda_\ell}\Psi$$

- We can manifest the magnetic 1-form symmetry via[1]

$$S[\tilde{A}] = -\int \mathrm{d}^4x \left(\frac{1}{4}\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} + (\partial_\mu + i\ell q\mathscr{A}_\mu)\Psi^*(\partial^\mu - i\ell q\mathscr{A}^\mu)\Psi + V(\Psi^*\Psi) - \frac{1}{2}\star\mathscr{F}^{\mu\nu}\tilde{A}_{\mu\nu}\right)$$

$$\tilde{A}_{\mu\nu} \to \tilde{A}_{\mu\nu} + \partial_\mu\tilde{\Lambda}_\nu - \partial_\nu\tilde{\Lambda}_\mu, \qquad \tilde{\mathscr{A}}_\mu \to \tilde{\mathscr{A}}_\mu - \tilde{\Lambda}_\nu$$

$$\star\mathscr{F}_{\mu\nu} = \partial_\mu\tilde{\mathscr{A}}_\nu - \partial_\nu\tilde{\mathscr{A}}_\mu$$

[1] Both symmetries cannot be gauged together on account of a mixed 't Hooft anomaly.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HIGHER-FORM SYMMETRIES

- In general spatial dimensions d, electromagnetism has an electric 1-form and magnetic $(d-2)$-form symmetry. The respective defects are free electric charges and magnetic monopoles.
- Electromagnetism can be viewed as a 1-form or $(d-2)$-form superfluid.
- Ordinary 0-form superfluids have a $(d-1)$-form symmetry, with the defects being vortices.
- Crystals also have a $(d-1)$-form symmetry, with the defects being dislocations.

HIGHER-FORM FLUIDS

with approximate higher-form symmetry

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THERMAL EQUILIBRIUM

- Many-body systems at thermal equilibrium can be characterised by their **thermal partition function**.
- Thermal partition function is a functional of background fields and can be used to obtain equilibrium values of (approximately) conserved densities and fluxes.
- For systems with spontaneously unbroken symmetries, the thermal partition function is a "local" functional of background fields.
- For systems with spontaneously broken symmetries, the thermal partition function is "non-local", and is given by a functional integral over the time-independent configurations of the Goldstone fields.

THERMAL EQUILIBRIUM: 0-FORM HYDROSTATICS

- For a 0-form symmetry, a thermal ensemble with constant charge density is described by a thermal partition function

$$\mathcal{Z}[A] = \exp \int \mathrm{d}^d x \left(\frac{1}{2} \chi \mu^2 + \ldots \right) \qquad \mu = \mu_0 + A_t$$

$$\implies \qquad J^t = n = \chi\mu + \ldots, \qquad J^i = 0$$

- This works because A_t is invariant under time-independent gauge transformations

$$A_t \to A_t + \partial_t \Lambda$$

This is no longer true for higher-form symmetries

$$A_{ti} \to A_{ti} + \partial_t \Lambda_i - \partial_i \Lambda_t$$

So, it is not possible to construct “local” partition functions with nonzero higher-form density.

THERMAL EQUILIBRIUM: 1-FORM HYDROSTATICS

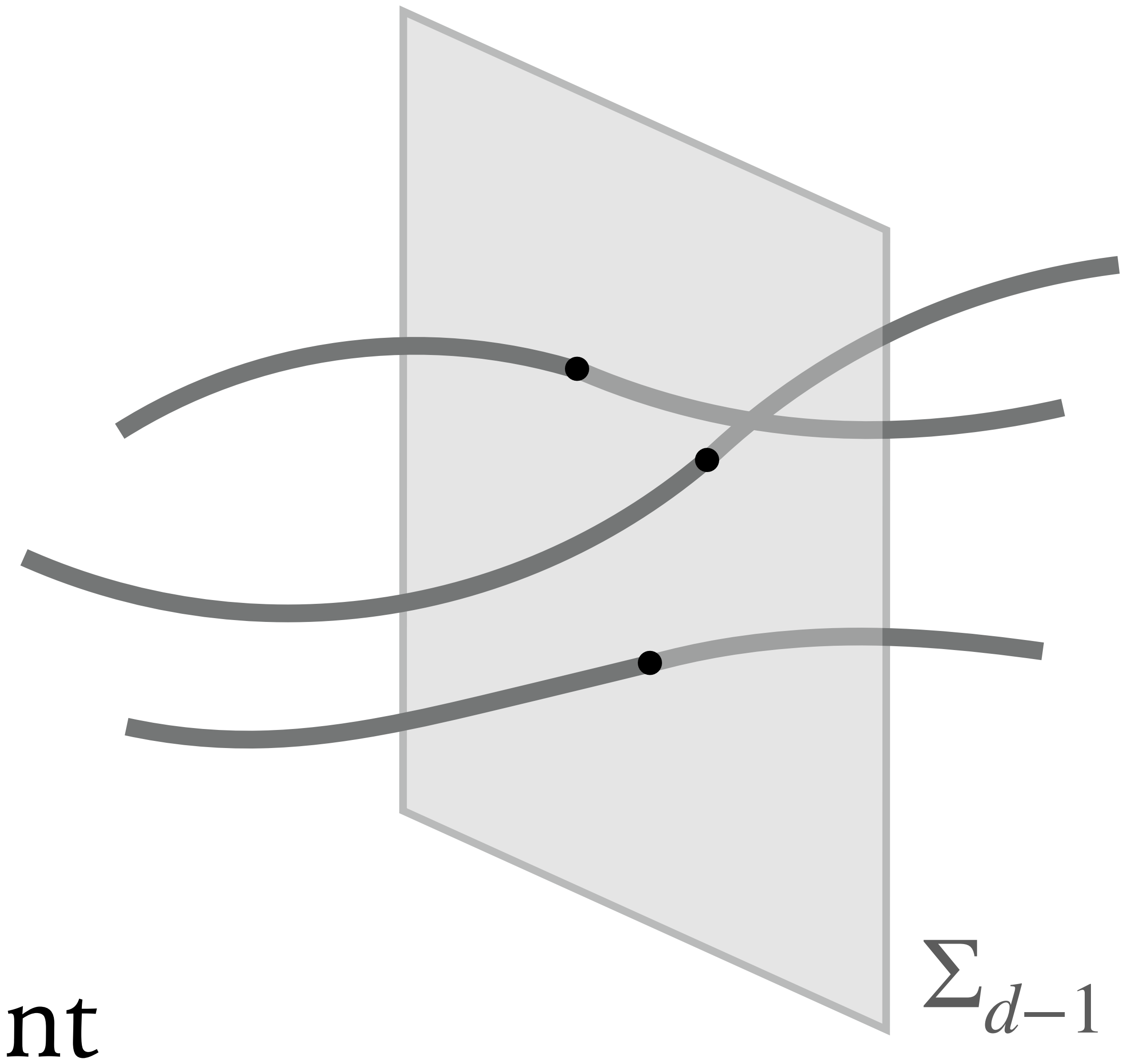
- We need to partially-spontaneously break the higher-form symmetry in the time-direction

$$\varphi \to \varphi - \Lambda_t \qquad\qquad \mu_i = -\,\partial_i\varphi + A_{ti}$$

This allows us to construct a "non-local" partition function

$$\mathscr{Z}[A] = \int \mathscr{D}\varphi \exp \int \mathrm{d}^3x \left(\frac{1}{2}\chi\,\mu_i\mu^i + \ldots\right)$$

$$\implies \qquad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = 0$$



- Classical configuration equation of φ implements the Gauss constraint

$$\partial_i J^{ti} = 0 \implies \partial_i\partial^i\varphi = 0 \qquad\qquad \varphi = -\,\mu_0 z \implies n_i = \chi\mu_0\,\delta_i^z$$

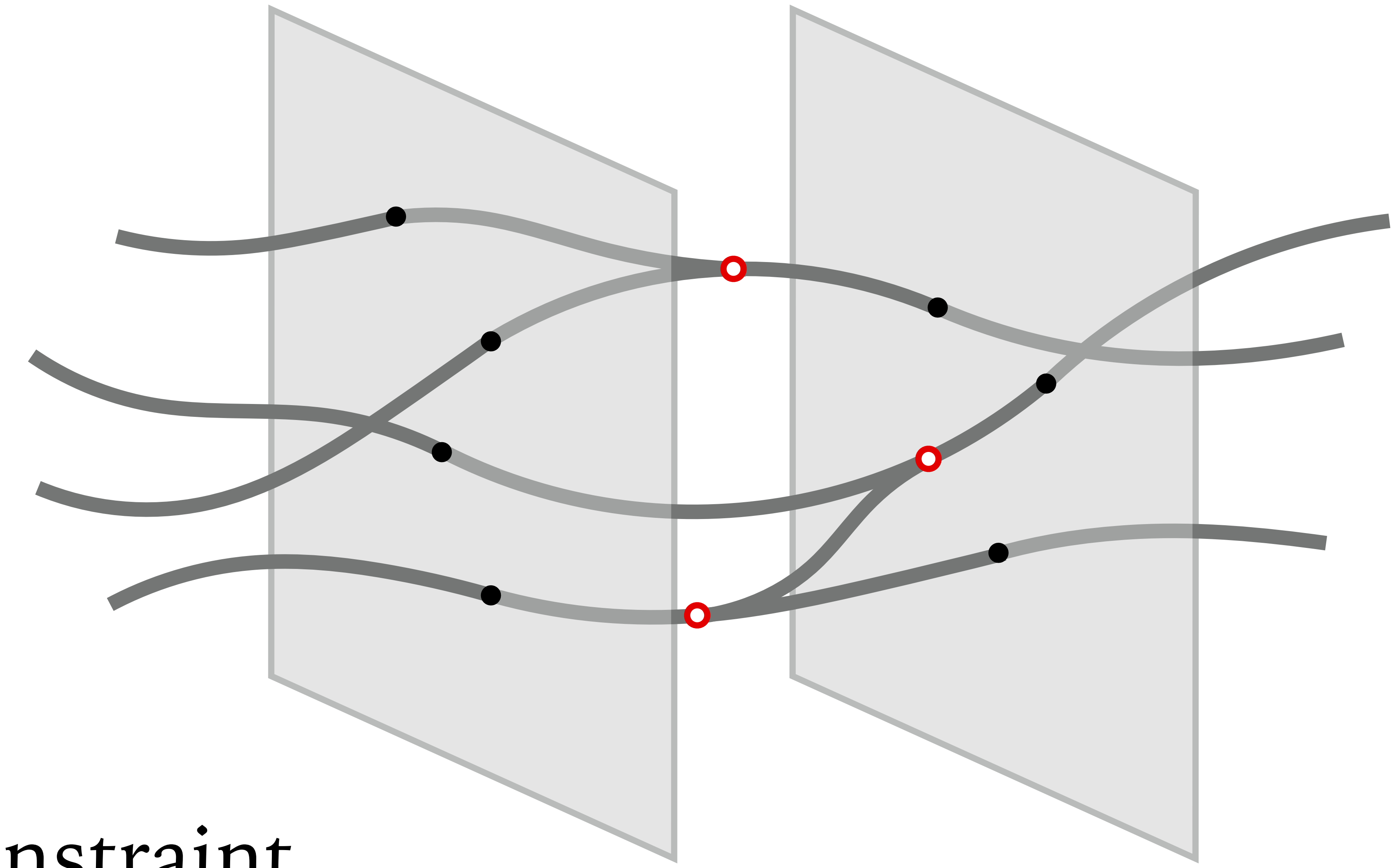
Armas, AJ [1808.01939, 1811.04913]

EXPLICITLY-BROKEN SYMMETRIES

- The partition function can also depend on Φ_μ through the "defect chemical potential"

$$\mu_\ell = -\ell\left(\varphi - \Phi_t\right) \qquad \Phi_t \to \Phi_t + \partial_t \Lambda_\ell$$

- The partition function takes the form



$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int \mathrm{d}^3x \left(\frac{1}{2}\chi\,\mu_i\mu^i + \frac{1}{2}\chi_\ell \mu_\ell^2 + \ldots \right)$$

$$\Longrightarrow \qquad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = 0$$

$$L^t = n_\ell = \chi_\ell \mu_\ell \qquad L^i = 0$$

- Classical configuration equation for φ imposes the Gauss constraint

$$\partial_i J^{ti} = \ell L^t \implies \partial_i \partial^i \varphi = k_0^2 \varphi \qquad \frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2 \chi_\ell}}$$

"Debye length"

Armas, AJ [2301.09628]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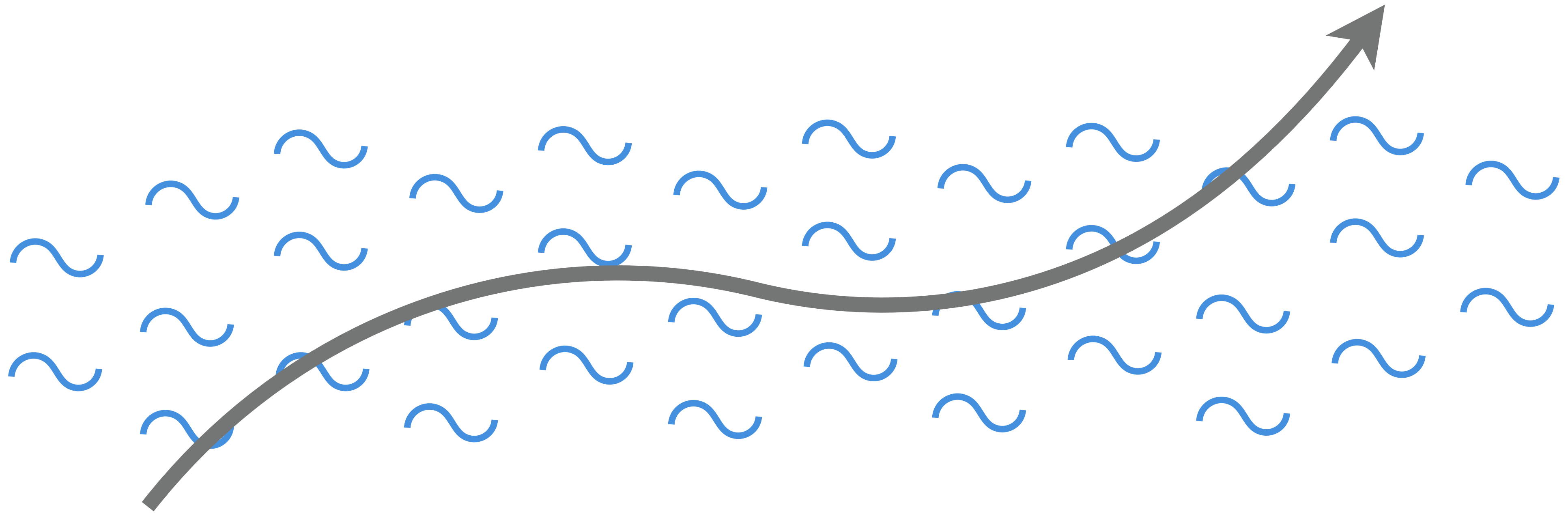
HYDRODYNAMICS

- **Hydrodynamics** is a framework to capture perturbative departures of a many-body system from thermal equilibrium.
- The relevant hydrodynamic degrees of freedom are a set of symmetry parameters corresponding to each global symmetry (conserved charge) of the system.
- Additionally, we need to add massless Goldstone fields for each spontaneously broken global symmetry.



0-FORM HYDRODYNAMICS

- The hydrodynamic description is based on the conservation laws

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$$

$$\nabla_\mu J^\mu = -\ell L, \qquad \nabla_\mu T^{\mu\nu} = F^{\nu\rho}J_\rho + \Xi^\mu L$$

$$\Xi_\mu = \partial_\mu \Phi + A_\mu$$

we have allowed for violation of the 0-form symmetry.

- The hydrodynamic fields are a set of symmetry parameters Λ_β, β^μ, transforming as

$$\delta\Lambda_\beta = \pounds_\chi \Lambda_\beta - \pounds_\beta \Lambda, \qquad \delta\beta^\mu = \pounds_\chi \beta^\mu$$

- These can be used to define gauge-invariant hydrodynamic fields μ, T, u^μ as

$$\frac{\mu}{T} = \Lambda_\beta + \beta^\mu A_\mu, \qquad \frac{u^\mu}{T} = \beta^\mu$$

0-FORM HYDRODYNAMICS

- Hydrodynamics is characterised by its constitutive relations

$$J^\mu[\mu, T, u^\mu; A_\mu, \Phi, g_{\mu\nu}], \qquad L[\mu, T, u^\mu; A_\mu, \Phi, g_{\mu\nu}], \qquad T^{\mu\nu}[\mu, T, u^\mu; A_\mu, \Phi, g_{\mu\nu}]$$

The constitutive relations are required to satisfy the second law of thermodynamics.

- For example, at first order in derivatives, we find the constitutive relations

$$J^\mu = n\, u^\mu - \sigma\, P^{\mu\nu} \left(T\partial_\nu \frac{\mu}{T} + u^\lambda F_{\lambda\nu} \right) \qquad P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$L = -\,\ell\, \sigma_\ell \left(u^\mu \Xi_\mu - \mu \right)$$

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p\, g^{\mu\nu} - \eta\, P^{\mu\rho} P^{\nu\sigma} \left(2\nabla_{(\rho} u_{\sigma)} - \frac{2}{d} P_{\rho\sigma} \nabla_\lambda u^\lambda \right) - \zeta\, P^{\mu\nu} \nabla_\lambda u^\lambda$$

- The coefficients follow the constraints

$$\delta p = s\delta T + n\delta\mu, \qquad \epsilon = Ts + \mu n - p, \qquad \sigma, \sigma_\ell, \eta, \zeta \geq 0$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the fluid sound and shear modes

$$v_s^2 = \frac{\partial p}{\partial \epsilon}$$

$$u_\parallel, T: \quad \omega = \pm v_s k - \frac{i}{2} D_\pi^\parallel k^2 + \ldots$$

$$u_\perp: \quad \omega = -iD_\pi^\perp k^2 + \ldots$$

$$D_\pi^\parallel = \frac{\zeta + 2\frac{d-1}{d}\eta}{\epsilon + p}, \qquad D_\pi^\perp = \frac{\eta}{\epsilon + p}$$

- The charge fluctuations give rise to a diffusive mode

$$\mu: \quad \omega = -iD_n k^2 - i\Gamma \qquad D_n = \frac{\sigma}{\chi}, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}$$

Charge fluctuations are damped due to explicit symmetry breaking.

1-FORM HYDRODYNAMICS

- The hydrodynamic description is based on the conservation laws

$$F_{\mu\nu\rho} = 3\partial_{[\mu}A_{\nu\rho]}$$

$$\Xi_{\mu\nu} = 2\partial_{[\mu}\Phi_{\nu]} + A_{\mu\nu}$$

$$\nabla_\mu J^{\mu\nu} = \ell L^\nu, \qquad \nabla_\mu L^\nu = 0, \qquad \nabla_\mu T^{\mu\nu} = \frac{1}{2}F^{\nu\rho\sigma}J_{\rho\sigma} + \ell\,\Xi^{\nu\rho}L_\rho$$

- The hydrodynamic fields are a set of symmetry parameters $\Lambda^\beta_\mu, \Lambda^\beta_\ell, \beta^\mu$, transforming as

$$\delta\Lambda^\beta_\mu = \pounds_\chi\Lambda^\beta_\mu - \pounds_\beta\Lambda_\mu, \qquad \delta\Lambda^\beta_\ell = \pounds_\chi\Lambda^\beta_\ell - \pounds_\beta\Lambda_\ell, \qquad \delta\beta^\mu = \pounds_\chi\beta^\mu$$

We also need the temporal Goldstone field φ transforming as

$$\delta\varphi = \pounds_\chi\varphi - \beta^\mu\Lambda_\mu$$

- These can be used to define the covariant hydrodynamic fields

$$\frac{\mu_\mu}{T} = \Lambda^\beta_\mu + \beta^\lambda A_{\lambda\mu} - \partial_\mu\varphi, \qquad \frac{\mu_\ell}{T} = -\,\ell\left(\varphi - \beta^\mu\Phi_\mu - \Lambda^\beta_\ell\right), \qquad \frac{u^\mu}{T} = \beta^\mu$$

JOSEPHSON EQUATION FOR TEMPORAL GOLDSTONE

- The dynamics of φ is governed by a Josephson equation of the form

$$\pounds_{\beta}\varphi = \beta^{\mu}\Lambda^{\beta}_{\mu} + \ldots$$

- We can absorb possible corrections to this equation by redefining Λ^{β}_{μ}. This implies

$$u^{\mu}\mu_{\mu} = u^{\mu}\Lambda^{\beta}_{\mu} - u^{\mu}\partial_{\mu}\varphi = 0$$

This still leaves redefinition freedom in the spatial components of Λ^{β}_{μ}.

GAUGE REDUNDANCY

- There is a gauge redundancy in the description that can be obtained by setting

$$\Lambda_\mu = \partial_\mu \lambda, \qquad \Lambda_\ell = \lambda$$

which leaves the background fields $A_{\mu\nu}, \Phi_\mu$ invariant.

- The dynamical fields transform as

$$\delta_\lambda \Lambda^\beta_\mu = -\partial_\mu \pounds_\beta \lambda, \qquad \delta_\lambda \Lambda^\beta_\ell = -\pounds_\beta \lambda, \qquad \delta_\lambda \beta^\mu = 0$$

$$\delta_\lambda \varphi = -\pounds_\beta \lambda$$

The physical hydrodynamic fields $\mu_\mu, \mu_\ell, T, u^\mu$ are invariant under these gauge transformations.

1-FORM HYDRODYNAMICS

- The constitutive relations are a straight-forward generalisation of the 0-form case

$$J^{\mu\nu} = 2u^{[\mu}n^{\nu]} - \sigma\, P^{\mu\rho}P^{\nu\sigma}\left(2T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} + u^{\lambda}F_{\lambda\rho\sigma}\right)$$

$$L^{\mu} = n_{\ell}u^{\mu} - \sigma_{\ell}P^{\mu\nu}\left(T\partial_{\nu}\frac{\mu_{\ell}}{T} + \ell u^{\lambda}\Xi_{\lambda\nu} - \ell\mu_{\nu}\right)$$

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} + p\, g^{\mu\nu} - n^{\mu}\mu^{\nu} - \eta\, P^{\mu\rho}P^{\nu\sigma}\left(2\nabla_{(\rho}u_{\sigma)} - \frac{2}{d}P_{\rho\sigma}\nabla_{\lambda}u^{\lambda}\right) - \zeta\, P^{\mu\nu}\nabla_{\lambda}u^{\lambda}$$

Note that σ_{ℓ} now behaves like a conductivity for the defect flux.

- The constraints are given as

$$\delta p = s\delta T + n^{\mu}\delta\mu_{\mu} + n_{\ell}\delta\mu_{\ell}, \qquad \epsilon = Ts + \mu_{\mu}n^{\mu} + \mu_{\ell}n_{\ell} - p, \qquad \sigma, \sigma_{\ell}, \eta, \zeta \geq 0$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu_\mu = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same fluid sound and shear modes.
- The charge fluctuations give rise to two diffusive modes

$$\mu_\perp : \quad \omega = -iD_n k^2 - i\Gamma$$

$$\mu_\| : \quad \omega = -iD_\ell k^2 - i\Gamma$$

$$D_n = \frac{\sigma}{\chi}, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}$$

$$D_\ell = \frac{\sigma_\ell}{\chi_\ell}$$

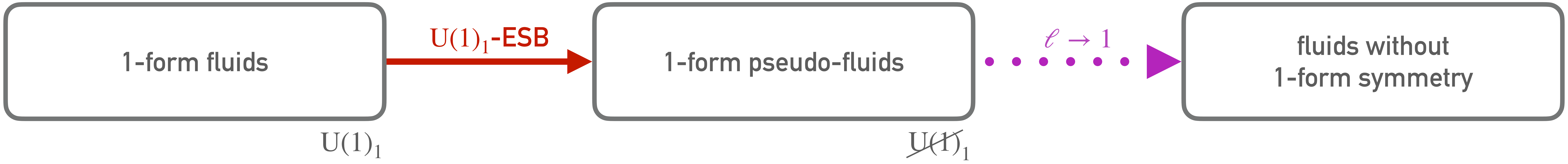
- The $\mu_\|$ mode obeys a damping-attenuation relation

$$\Gamma = D_\ell \, k_0^2$$

$$k_0^2 = \frac{\ell^2 \chi_\ell}{\chi}$$

TOPOLOGICAL PHASE TRANSITIONS

- If we increase the strength of defects, i.e. increase ℓ, the charge fluctuations gap out and we are left with a fluid without 1-form symmetry.



1-form fluids

$\mathrm{U}(1)_1$

$\mathrm{U}(1)_1$-ESB

1-form pseudo-fluids

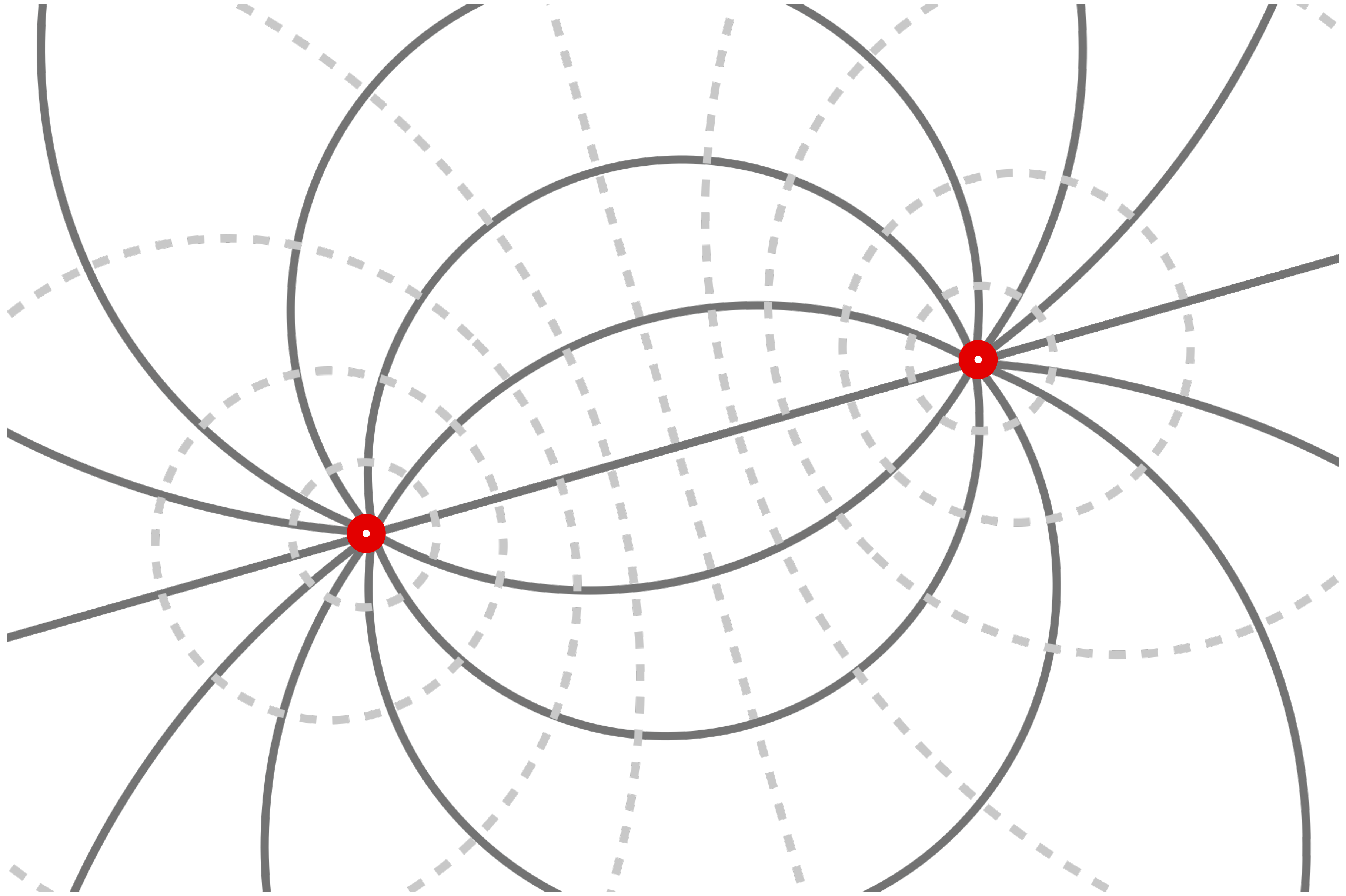
$\cancel{\mathrm{U}(1)}_1$

$\ell \to 1$

fluids without
1-form symmetry

$$\nabla_\mu\left(u^\mu n^\nu - u^\nu n^\mu + \ldots\right) = \ell n_\ell u^\mu + \Gamma n^\mu + \ldots$$

- The same discussion also applies to the phase transition of 0-form superfluids to 0-form fluids, mediated by vortices.



HIGHER-FORM SUPERFLUIDS

with approximate higher-form symmetry

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1-FORM SUPERFLUIDS IN EQUILIBRIUM

- In the superfluid phase, the higher-form symmetry is completely spontaneously-broken

$$\varphi \to \varphi - \Lambda_t \qquad \mu_i = -\partial_i \varphi + A_{ti}$$

$$\phi_i \to \phi_i - \Lambda_i \qquad \xi_{ij} = \partial_i \phi_j - \partial_j \phi_i + A_{ij}$$

- The partition function takes the form

$$\mathscr{Z}[A] = \int \mathscr{D}\varphi \mathscr{D}\phi_i \exp \int \mathrm{d}^3 x \left(\frac{1}{2} \chi\, \mu_i \mu^i - \frac{1}{4\tilde{\chi}} \xi_{ij} \xi^{ij} + \ldots \right)$$

$$\Longrightarrow \qquad J^{ti} = n^i = \chi\, \mu^i, \qquad J^{ij} = -\frac{1}{\tilde{\chi}} \xi^{ij}$$

- The configuration equations imply

$$\partial_i J^{ti} = 0 \implies \partial_i \partial^i \phi = 0 \qquad \varphi = -\mu_0 z \implies n_i = \chi \mu_0\, \delta_i^z$$

$$\partial_t J^{ti} + \partial_k J^{ki} = 0 \implies \partial_k \partial^k \phi^i - \partial^i \partial_k \phi^k = 0 \qquad \phi^i = (0,\, \tilde{\mu}_0 x,\, 0) \implies \xi_{ij} = \tilde{\mu}_0\, \varepsilon_{ijz}$$

Armas, AJ [1811.04913]

1-FORM PSEUDO-SUPERFLUIDS IN EQUILIBRIUM

- In the presence of explicit symmetry breaking, the 1-form superfluid can exist in two phases depending on the 0-form defect symmetry being spontaneously broken or not.
- In the **relaxed/Coulomb phase**, the 0-form defect symmetry is spontaneously unbroken and we can only construct the "defect chemical potential"

$$\mu_\ell = -\ell \left(\varphi - \Phi_t\right)$$

- In the **pinned/Higgs phase**, the 0-form defect symmetry is spontaneously broken

$$\phi_\ell \to \phi_\ell - \Lambda_\ell$$

This allows us to also construct the phase misalignment vector

$$\psi_i = \ell \left(\phi_i - \Phi_i - \partial_i \phi_\ell\right)$$

In the language of Higgs mechanism, the 1-form phase ϕ_i can eat 0-form phase ϕ_ℓ to become massive.

Armas, AJ [2301.09628]

HIGHER-FORM PSEUDO-SUPERFLUIDS IN EQUILIBRIUM

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int \mathrm{d}^3x \left(\frac{1}{2}\chi\, \mu_i\mu^i - \frac{1}{4\tilde{\chi}}\xi_{ij}\xi^{ij} + \frac{1}{2}\chi_\ell \mu_\ell^2 - \frac{m^2}{2}\psi_i\psi^i + \dots \right)$$

$$\implies \qquad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = -\frac{1}{\tilde{\chi}}\xi^{ij}$$

$$L^t = n_\ell = \chi_\ell \mu_\ell \qquad L^i = m^2\psi^i$$

- Classical configuration equation for φ, ϕ_i impose the conservation equations in equilibrium

$$\partial_i J^{ti} = \ell L^t \implies \partial_i\partial^i\varphi = k_0^2\varphi$$

$$\partial_t J^{ti} + \partial_k J^{ki} = -\ell L^i \implies \partial_k\partial^k\phi^i - \partial^i\partial_k\phi^k = k_{0\phi}^2\phi^i$$

$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2\chi_\ell}}$$

"Debye length"

$$\frac{1}{k_{0\phi}} = \sqrt{\frac{1}{\ell^2 m^2\tilde{\chi}}}$$

"London depth"

Armas, AJ [2301.09628]

1-FORM PSEUDO-SUPERFLUID DYNAMICS

- The conservation equations for a 1-form superfluid remain the same

$$\nabla_\mu J^\mu = -\ell L, \qquad \nabla_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho + \Xi^\mu L$$

- We have an additional Josephson equation for the Goldstone phase ϕ_μ

$$\pounds_\beta \phi_\mu = \Lambda^\beta_\mu + \frac{1}{T}\mathscr{K}_\mu \quad \Longrightarrow \quad u^\mu \xi_{\mu\nu} = \mu_\nu + \mathscr{K}_\nu$$

$$\xi_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$$

$$\beta^\mu \phi_\mu = \varphi$$

$$u^\mu \mathscr{K}_\mu = 0$$

- In the pinned/Higgs phase, we also have a Josephson equation for the phase ϕ_ℓ

$$\pounds_\beta \phi_\ell = \Lambda^\beta_\ell + \frac{1}{T}\mathscr{K}_\ell \quad \Longrightarrow \quad u^\mu \psi_\mu = -\mu_\ell - \ell \mathscr{K}_\ell$$

$$\psi_\mu = -\ell\left(\phi_\mu - \Phi_\mu\right)$$

1-FORM PSEUDO-SUPERFLUID DYNAMICS

- The constitutive relations are given as

$$J^{\mu\nu} = 2u^{[\mu}n^{\nu]} - \zeta^{\mu\nu} - \sigma\, P^{\mu\rho}P^{\nu\sigma}\left(2T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} + u^{\lambda}F_{\lambda\rho\sigma}\right)$$

$$\zeta^{\mu\nu} = \frac{1}{\tilde{\chi}}P^{\mu\rho}P^{\nu\sigma}\xi_{\rho\sigma}$$

$$L^{\mu} = n_{\ell}u^{\mu} + m^{2}\bar{\psi}^{\mu} - \sigma_{\ell}\, P^{\mu\nu}\left(T\partial_{\nu}\frac{\mu_{\ell}}{T} + \ell\, u^{\lambda}\Xi_{\lambda\nu} - \ell\,\mu_{\nu}\right) - \gamma\, P^{\mu\nu}\nabla^{\lambda}\zeta_{\lambda\nu}$$

$$\bar{\psi}^{\mu} = P^{\mu\nu}\psi_{\nu}$$

$$u^{\lambda}\xi_{\lambda\mu} = \mu_{\mu} - \tilde{\sigma}\, P^{\mu\nu}\nabla^{\lambda}\zeta_{\lambda\nu} - \gamma\, P^{\mu\nu}\left(T\partial_{\nu}\frac{\mu_{\ell}}{T} + \ell\, u^{\lambda}\Xi_{\lambda\nu} - \ell\,\mu_{\nu}\right)$$

$$u^{\mu}\psi_{\mu} = -\mu_{\ell} + \ell\,\tilde{\sigma}_{\psi}\nabla_{\mu}\left(m^{2}\bar{\psi}^{\mu}\right)$$

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} + p\, g^{\mu\nu} - \chi\,\mu^{\mu}\mu^{\nu} + \tilde{\chi}\,\zeta^{\mu\rho}\zeta^{\nu}{}_{\rho}$$
$$-\eta\, P^{\mu\rho}P^{\nu\sigma}\left(2\nabla_{(\rho}u_{\sigma)} - \frac{2}{d}P_{\rho\sigma}\nabla_{\lambda}u^{\lambda}\right) - \zeta\, P^{\mu\nu}\nabla_{\lambda}u^{\lambda}$$

1-FORM PSEUDO-SUPERFLUID DYNAMICS

- The transport coefficients follow the constraints

$$\delta p = s\delta T + n^{\mu}\delta\mu_{\mu} + \frac{1}{2}\zeta^{\mu\nu}\delta\xi_{\mu\nu} + n_{\ell}\delta\mu_{\ell}, \qquad \epsilon = Ts + \mu_{\mu}n^{\mu} + \mu_{\ell}n_{\ell} - p$$

$$\sigma, \tilde{\sigma}_{\psi}, \eta, \zeta \geq 0, \qquad \tilde{\sigma}\,\sigma_{\ell} \geq \gamma^{2}$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu_\mu = \xi_{\mu\nu} = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same fluid sound and shear modes.

- The transverse charge and Goldstone fluctuations give rise to the photon mode

$$\mu_\perp, \phi_\perp : \quad \left(i\omega - D_n k^2 - \Gamma\right)\left(i\omega - \tilde{D}_n k^2 - \tilde{\Omega}\right) + \omega_0^2 + v_\perp^2 k^2 = 0$$

$$v_\perp^2 = \frac{\lambda^2}{\chi\tilde{\chi}}, \qquad \omega_0^2 = \frac{\ell^2 m^2 \lambda^2}{\chi}$$

$$D_n = \frac{\sigma}{\chi}, \qquad \tilde{D}_n = \frac{\tilde{\sigma}}{\tilde{\chi}}, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}, \qquad \tilde{\Omega} = \ell^2 m^2 \tilde{\sigma}$$

- Damping-attenuation relation

$$\tilde{\Omega} = \tilde{D}_n k_{0\phi}^2 \qquad\qquad k_{0\phi}^2 = \ell^2 m^2 \tilde{\chi}$$

LINEARISED FLUCTUATIONS

- The longitudinal charge and Goldstone fluctuations give rise to another mode

$$\mu_\parallel, \phi_\parallel : \quad (i\omega - D_\ell k^2 - \Gamma)\left(i\omega - \tilde{D}_\psi k^2 - \tilde{\Omega}\right) + \omega_0^2 + v_\parallel^2 k^2 = 0$$

$$v_\parallel^2 = \frac{\lambda^2 m^2}{\chi_\ell}, \qquad \omega_0^2 = \frac{\ell^2 m^2 \lambda^2}{\chi}$$

$$D_\ell = \frac{\sigma_\ell}{\chi_\ell}, \qquad \tilde{D}_\psi = \ell^2 m^2 \tilde{\sigma}_\psi, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}, \qquad \tilde{\Omega} = \ell^2 m^2 \tilde{\sigma}$$

- Damping-attenuation relation

$$\Gamma = D_\ell k_0^2 \qquad\qquad k_0^2 = \frac{\ell^2 \chi_\ell}{\chi}$$

TOPOLOGICAL PHASE TRANSITIONS

- We can implement topological phase transitions by increasing the strength of ℓ, thereby increasing the strength of explicit symmetry breaking.
- The product of the phase transition depends on if we are starting form the relaxed/Coulomb phase of pinned/Higgs phase.
- In the relaxed/Coulomb phase, we arrive at a $(d-2)$-form fluid. In the context of electromagnetism, this describes magnetohydrodynamics with conserved magnetic field lines.

$$\omega = -i\left(\tilde{D}_n + \frac{v_\perp^2}{\Gamma}\right)k^2$$

- In the pinned/Higgs phase, we arrive at a neutral fluid. In the context of electromagnetism, this describes expulsion of all electromagnetic fields inside a superconductor.


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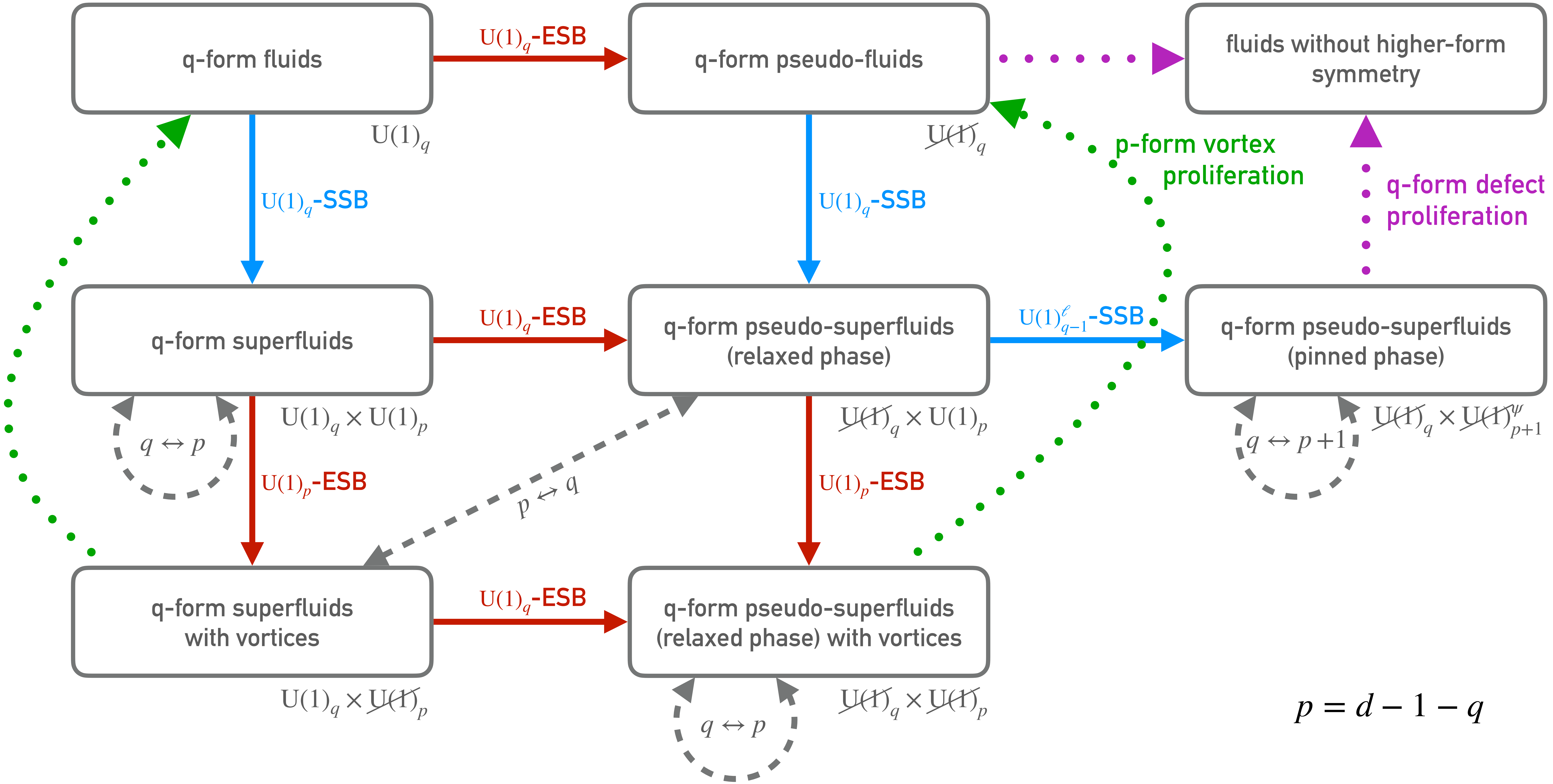
HIGHER-FORM SUPERFLUIDS

- It is possible to also keep the "magnetic" $(d-2)$-form symmetry of a 1-form superfluid manifest by coupling the system to a $(d-1)$-form gauge field and accounting for the mixed anomaly.
- Explicit breaking of the "magnetic" $(d-2)$-form symmetry give rise to vortices in the 1-form superfluid. Only relaxed/ Coulomb phase can admit vortices.
- Extension to q-form superfluids is straight-forward.



q-form fluids

$\mathrm{U}(1)_q$-ESB

q-form pseudo-fluids

fluids without higher-form symmetry

$\mathrm{U}(1)_q$

$\cancel{\mathrm{U}(1)}_q$

$\mathrm{U}(1)_q$-SSB

$\mathrm{U}(1)_q$-SSB

p-form vortex proliferation

q-form defect proliferation

q-form superfluids

$\mathrm{U}(1)_q$-ESB

q-form pseudo-superfluids (relaxed phase)

$\mathrm{U}(1)^{\ell}_{q-1}$-SSB

q-form pseudo-superfluids (pinned phase)

$q \leftrightarrow p$

$\mathrm{U}(1)_q \times \mathrm{U}(1)_p$

$\cancel{\mathrm{U}(1)}_q \times \mathrm{U}(1)_p$

$q \leftrightarrow p+1$

$\cancel{\mathrm{U}(1)}_q \times \cancel{\mathrm{U}(1)}^{\psi}_{p+1}$

$\mathrm{U}(1)_p$-ESB

$p \leftrightarrow q$

$\mathrm{U}(1)_p$-ESB

q-form superfluids with vortices

$\mathrm{U}(1)_q$-ESB

q-form pseudo-superfluids (relaxed phase) with vortices

$\mathrm{U}(1)_q \times \cancel{\mathrm{U}(1)}_p$

$q \leftrightarrow p$

$\cancel{\mathrm{U}(1)}_q \times \cancel{\mathrm{U}(1)}_p$

$$p = d - 1 - q$$

OUTLOOK

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OUTLOOK

- Higher-form symmetries can be used to classify phases of matter with topological order.
- Breaking of continuous higher-form symmetries is associated with topological defects, which mediate topological phase transitions.
- A hydrodynamic theory with approximate higher-form symmetries provides a model for dynamical phase transitions based on symmetries.
- Further applications include emergent magnetic monopoles in spin ice, plasma phase transitions, melting phase transition in higher-dimensions, superfluid and superconductor phase transitions.

Thank You