

STABLE AND CAUSAL RELATIVISTIC SCHWINGER-KELDysh HYDRODYNAMICS

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© Phil Smith "Stochastic 2"



OVERVIEW

- Schwinger-Keldysh Hydrodynamics¹ is a “better” way to do Hydrodynamics.
 - **Effective action / path-integral description:** cleaner to implement symmetries.
 - **Systematic inclusion of stochastic noise:** allows access to far-from-equilibrium physics, e.g. critical points, active matter.
 - **Built in: second law of thermodynamics, Onsager’s relations, (non-linear) fluctuation-dissipation theorems (FDTs).**
 - **Extends the applicability** of hydrodynamics: non-universal stochastic interactions and quantum interactions.
 - **Connections with quantum chaos.**
- Today’s discussion: how to circumvent the **stability and causality issues** in relativistic Schwinger-Keldysh hydrodynamics.

¹[Crossley, Glorioso, Liu [1511.03646]] [Haehl, Loganayagam, Rangamani [1511.07809]]
[Jensen, Pinzani-Fokeeva, Yarom [1701.07436]]



LESSONS

- Low-energy description of a fluid, classical or stochastic, is **not** completely characterised by its constitutive relations. New **stochastic transport coefficients**¹ characterising 4- and higher-point stochastic interactions “contaminate” hydrodynamic observables at higher derivative orders.
- Ensuring the 2-point FDT ($G^S \propto \text{Im } G^R$) in Martin-Siggia-Rose (MSR) formalism (stochastic hydrodynamics) **does not** guarantee consistency with higher-point FDTs.²
- Not all “stable and causal formalisms” of relativistic hydrodynamics are created equal: *Müller-Israel-Stewart Hydrodynamics* is consistent with stochastic fluctuations, while the newer *Bemfica-Disconzi-Noronha-Kovtun hydrodynamics*⁴ **is not**.^{2,3}
- Consistent UV-completions of hydrodynamics, like Müller-Israel-Stewart Hydrodynamics, may simultaneously realise **more than one** Kubo-Martin-Schwinger (KMS) symmetries.¹

¹[AJ, Kovtun [2009.01356]]

²[AJ, Kovtun [2309.00511]], ³[Mullins, Hippert, Gavassino, Noronha [2309.00512]]

⁴[Kovtun [1907.08191]] [Bemfica, Disconzi, Noronha [1907.12695]]

CLASSICAL HYDRODYNAMICS (REALLY, DIFFUSION)

- Conservation equations associated with the global symmetries (e.g., U(1))

$$\partial_\mu J^\mu = 0$$

- Source A_μ coupled to J^μ . Background “gauge” transformations:

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

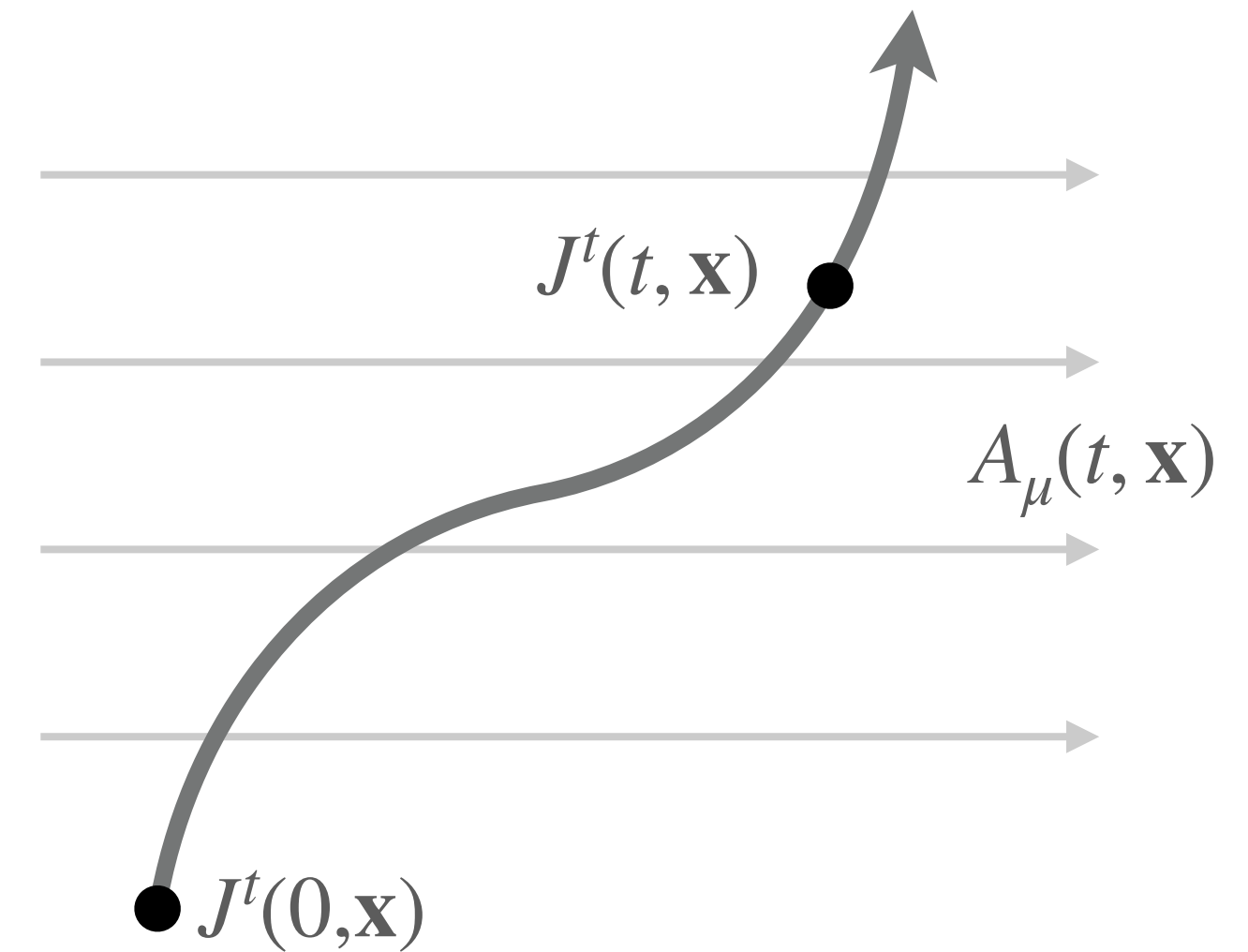
- Constitutive relations:

$$J^\mu[\mu, A_\mu] = n(\mu) u^\mu - D(\mu) \Delta^{\mu\nu} \partial_\nu n(\mu) + \sigma(n) F^{\mu\nu} u_\nu + \dots \text{higher derivatives} \dots$$

- Second law: KMS/FDT condition : $\chi D = \sigma$, Positivity : $\sigma \geq 0$

- Retarded correlators:

$$G_{JJ\dots}^R = \left(\frac{\delta}{\delta A} \frac{\delta}{\delta A} \dots \right) J_{\text{onshell}}[A] \Big|_{\delta A=0}$$



$$u^\mu u_\mu = -1$$

$$\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\chi = \frac{\partial n}{\partial \mu}$$

CLASSICAL HYDRODYNAMICS (AGAIN, DIFFUSION)

► Effective field theory: *dynamical fields* $\varphi_{r,a}$, *background fields* $A_{r,a\mu}$

$$B_{r,a\mu} \equiv \partial_\mu \varphi_{r,a} + A_{r,a\mu}$$

$$S = \int_x n(\mu) u^\mu \mathbf{B}_{a\mu} + iT\sigma(\mu) \Delta^{\mu\nu} \mathbf{B}_{a\mu} \left(\mathbf{B}_{a\nu} + i\mathfrak{L}_\beta B_{r\nu} \right)$$

$$\beta^\mu \equiv \frac{u^\mu}{T}, \quad \mathfrak{L}_\beta \equiv \beta^\mu \partial_\mu + \dots$$

$$\mu \equiv u^\mu B_{r\mu}$$

$$J_{r,a}^\mu = \frac{\delta S}{\delta A_{a,r}}$$

$$J_r^\mu = J_{\text{classical}}^\mu + \text{noise-}\mathcal{O}(\varphi_a, A_a)$$

$$\varphi_{r,a}\text{-EOM : } \partial_\mu J_{a,r}^\mu = 0$$

$$J_a^\mu = \text{noise-}\mathcal{O}(\varphi_a, A_a)$$

SK Conditions

$$f_a \rightarrow 0 \implies S \rightarrow 0$$

$$f_a \rightarrow -f_a \implies S \rightarrow -S^*$$

$$\text{Im } S \geq 0.$$

KMS Symmetry

$$f_r \leftrightarrow \Theta f_r$$

$$f_a \leftrightarrow \Theta \left(f_a + i\mathfrak{L}_\beta f_r \right)$$

$$\Theta = \text{T, PT, CT, or CPT}$$

Global Symmetries

$$\varphi_{r,a} \rightarrow \varphi_{r,a} - \Lambda_{r,a}$$

$$A_{r,a} \rightarrow A_{r,a} + \partial \Lambda_{r,a}$$

Diagonal Shift Symmetry

$$\varphi_r \rightarrow \varphi_r + \lambda(\mathbf{x})$$

STABILITY AND CAUSALITY ISSUES

- Two-point functions in a **rest** frame of reference

$$G_{J_t J_t}^R(p) = \frac{ik^2 \sigma}{\omega + iDk^2} \qquad G_{J_t J_t}^S(p) = 2T \frac{k^2 \sigma}{|\omega + iDk^2|^2}$$

- In a **boosted** frame of reference, the denominator of G^R modifies to

$$-u^\mu p_\mu + iD\Delta^{\mu\nu} p_\mu p_\nu$$

$$u^\mu = \frac{1}{\sqrt{1 - \mathbf{v}^2}}(1, \mathbf{v})$$

- There is an poles in the “upper-half complex- ω plane” — **unstable**.

$$\omega = i \frac{\sqrt{1 - \mathbf{v}^2}}{\mathbf{v}^2 D} + \mathcal{O}(k)$$

- Poles propagate faster than the speed of light for large- k — **acausal**.

$$\omega = \left(\frac{1}{|\mathbf{v}|} + \mathcal{O}(1/k) \right) k$$

BDNK-DIFFUSION

- BDNK model of diffusion¹

$$J^\mu = \left(n + \tau u^\mu \partial_\mu n \right) - \sigma \Delta^{\mu\nu} \left(\partial_\nu \mu - F^{\mu\nu} u_\nu \right)$$

- UV-regularisation: $\omega = -i \frac{\sqrt{1 - \mathbf{v}^2}}{\tau - \mathbf{v}^2 D} + \mathcal{O}(k) \quad \omega = \left(\frac{(\tau - D) |\mathbf{v}|}{\tau - \mathbf{v}^2 D} + \mathcal{O}(1/k) \right) k$

- Effective action:

$$S = \int_x n u^\mu B_{a\mu} - iT \chi \tau u^\mu u^\nu B_{a\mu} \left(B_{a\nu} + i \mathfrak{L}_\beta B_{r\nu} \right) + iT \sigma \Delta^{\mu\nu} B_{a\mu} \left(B_{a\nu} + i \mathfrak{L}_\beta B_{r\nu} \right)$$

- Violates either stability or $\text{Im } S \geq 0$.

$$G_{J_t J_t}^R(p) = \frac{(1 - i\omega\tau) i k^2 \sigma}{\omega(1 - i\omega\tau) + i D k^2}$$

$$G_{J_t J_t}^S(p) = 2T \frac{(1 + \omega^2 \tau^2) k^2 \sigma - \chi \tau D^2 k^4}{|\omega(1 - i\omega\tau) + i D k^2|^2}$$

¹[Kovtun [1907.08191]] [Bemfica, Disconzi, Noronha [1907.12695]]

IMPROVED BDNK-DIFFUSION

- BDNK model of diffusion

$$J^\mu = \left(n + \tau u^\mu \partial_\mu n \right) - \sigma \Delta^{\mu\nu} \left(\partial_\nu \mu - F^{\mu\nu} u_\nu \right)$$

- UV-regularisation: $\omega = -i \frac{\sqrt{1 - \mathbf{v}^2}}{\tau - \mathbf{v}^2 D} + \mathcal{O}(k) \quad \omega = \left(\frac{(\tau - D) |\mathbf{v}|}{\tau - \mathbf{v}^2 D} + \mathcal{O}(1/k) \right) k$

- Improved effective action:

$$S = \int_x n u^\mu B_{a\mu} - iT \chi \tau u^\mu u^\nu B_{a\mu} \left(i \mathfrak{L}_\beta B_{r\nu} \right) + iT \sigma \Delta^{\mu\nu} B_{a\mu} \left(B_{a\nu} + i \mathfrak{L}_\beta B_{r\nu} \right)$$

- Does not realise KMS symmetry — violate FDT for 3-point correlation functions when $\partial\sigma/\partial n \neq 0$.

MAXWELL-CATTANEO DIFFUSION

- Maxwell-Cattaneo model of diffusion

$$J^\mu = nu^\mu + \mathcal{J}^\mu$$

$$\tau \Delta_\nu^\mu u^\lambda \partial_\lambda \mathcal{J}^\nu + \mathcal{J}^\mu = -\sigma \Delta^{\mu\nu} (\partial_\nu \mu - F^{\mu\nu} u_\nu)$$

- Effective action:

$$S = \int_x nu^\mu B_{a\mu} + \Delta^{\mu\nu} (\mathbf{B}_{a\mu} - \mathcal{J}_{a\mu}) \cdot \mathcal{J}_\nu + iT\sigma \Delta^{\mu\nu} \mathcal{J}_{a\mu} \cdot \left(\mathcal{J}_{a\nu} + \left(i\mathfrak{L}_\beta B_{r\nu} + \frac{\tau}{\sigma} i\mathfrak{L}_\beta \mathcal{J}_\nu \right) \right)$$

- Realises two KMS symmetries

$$\mathcal{J} \leftrightarrow \mathcal{J} \Big|_{-x}$$

(a)

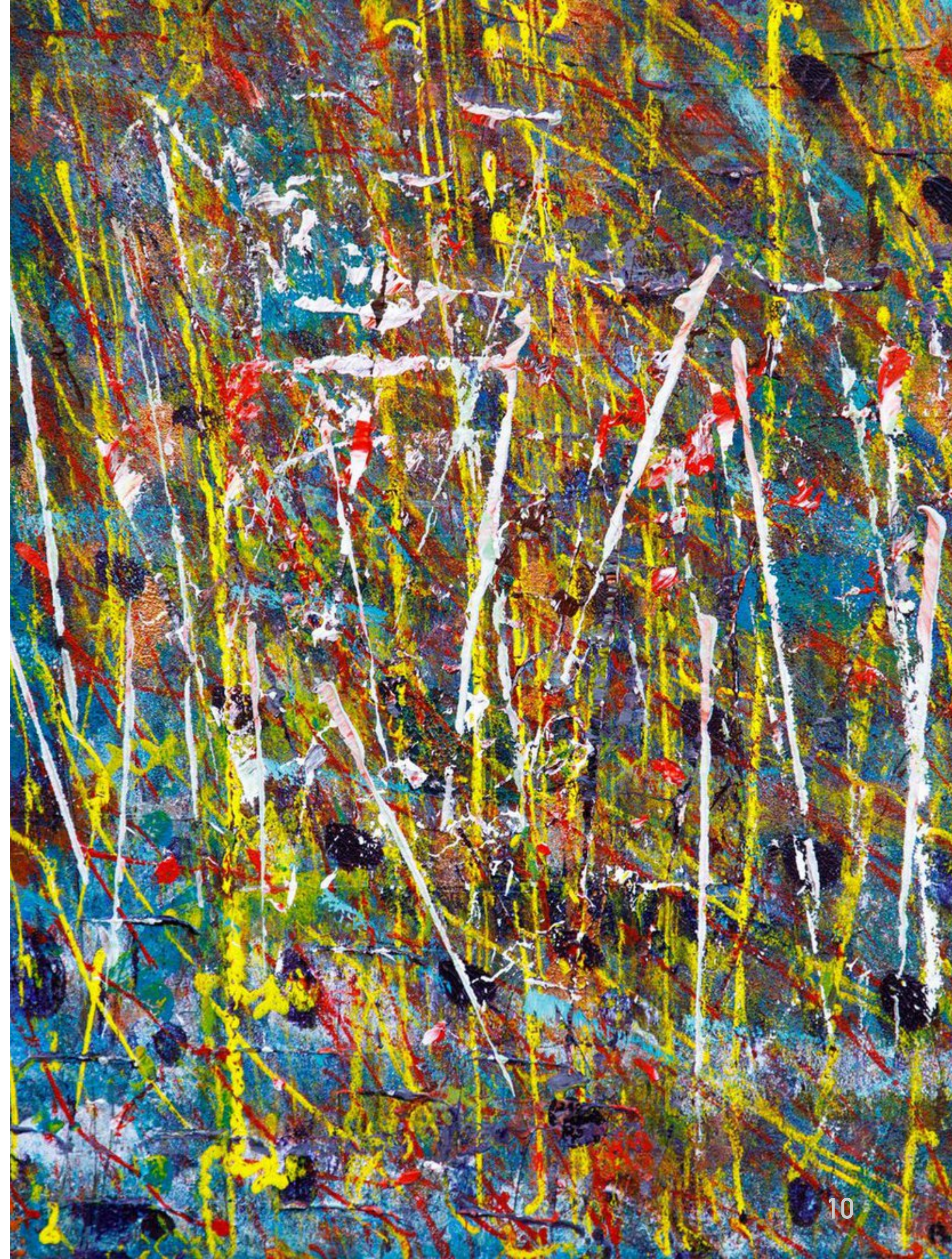
$$\mathcal{J}_a \leftrightarrow \mathcal{J}_a + i \left(\mathfrak{L}_\beta B_r + \frac{\tau}{\sigma} \mathfrak{L}_\beta \mathcal{J} \right) \Big|_{-x}$$

or

$$\mathcal{J}_a \leftrightarrow -\mathcal{J}_a - \frac{i}{T\sigma} \mathcal{J} \Big|_{-x}$$

(b)

BACKUP SLIDES





THE BIG PICTURE

- **Effective field theory** is a robust framework for describing physical many-body systems without the detailed knowledge of their microscopic constituents.
- The conventional treatment of effective field theory is only suited for **zero temperature** or equilibrium phenomena at **finite temperature** (statistical field theory).
- **Hydrodynamics** is a “*universal*” low-energy effective description based on conservation laws that allows us to perturbatively depart from thermal equilibrium.
- Classical hydrodynamics has limited applicability due to the presence of **stochastic thermal fluctuations**.
- **Schwinger-Keldysh hydrodynamics** is a *symmetry-based effective field theory* generalisation of classical hydrodynamics that aims to systematically incorporate the effects of arbitrary stochastic thermal fluctuations.



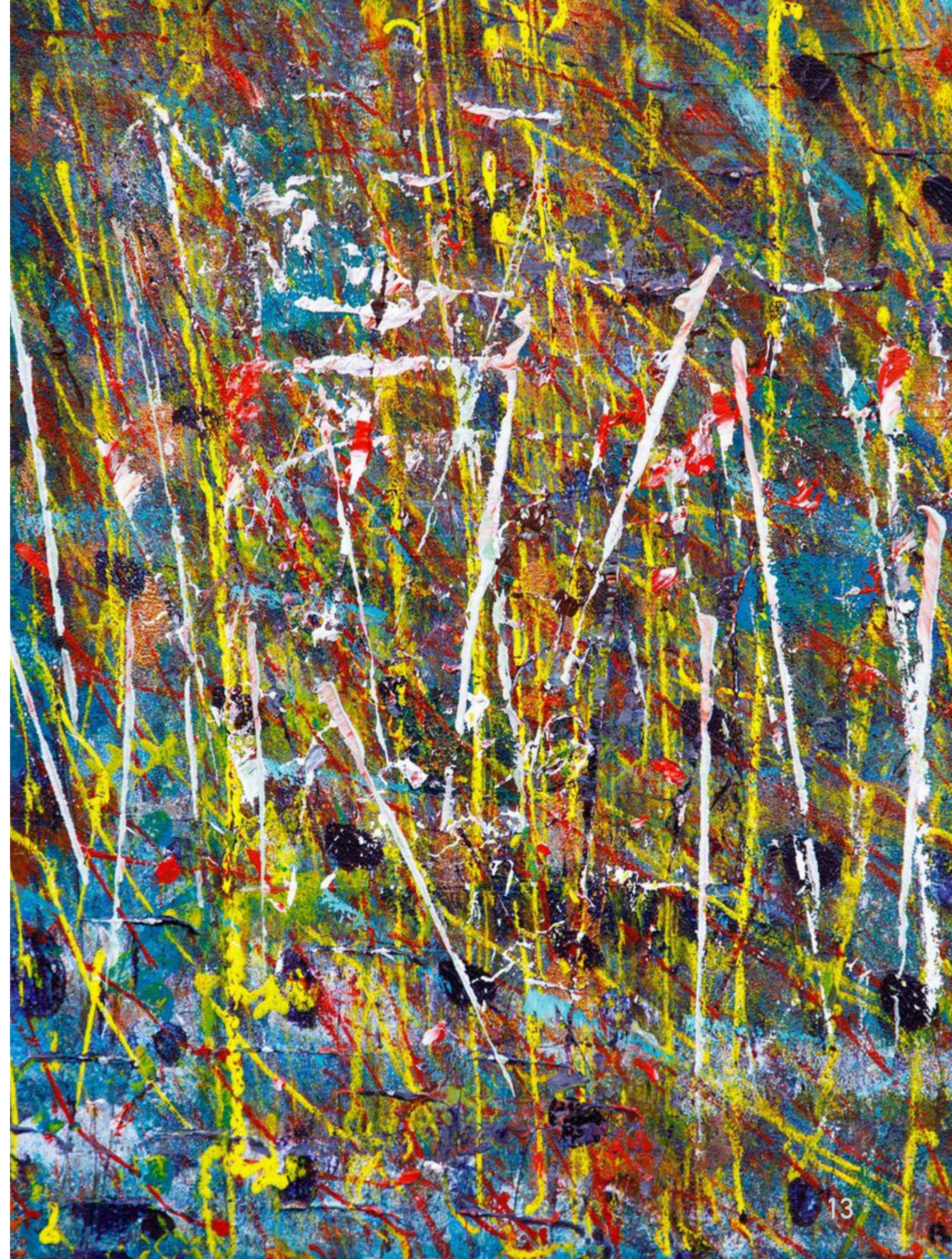
OUTLINE

- Classical hydrodynamics
- Stochastic hydrodynamics and beyond¹
- Schwinger-Keldysh framework¹
- Schwinger-Keldysh with non-conserved variables²

¹[AJ, Kovtun [2009.01356]] || ²[AJ, Kovtun, [2309.00511]]

CLASSICAL HYDRODYNAMICS

a sketch



CLASSICAL HYDRODYNAMICS (REALLY, DIFFUSION)

- The dynamical equations are **conservation equations** associated with the global symmetries.
For example, U(1) symmetry

$$\partial_\mu J^\mu = 0$$

This equation determines how the *density* $J^t = n$ evolves in time.*

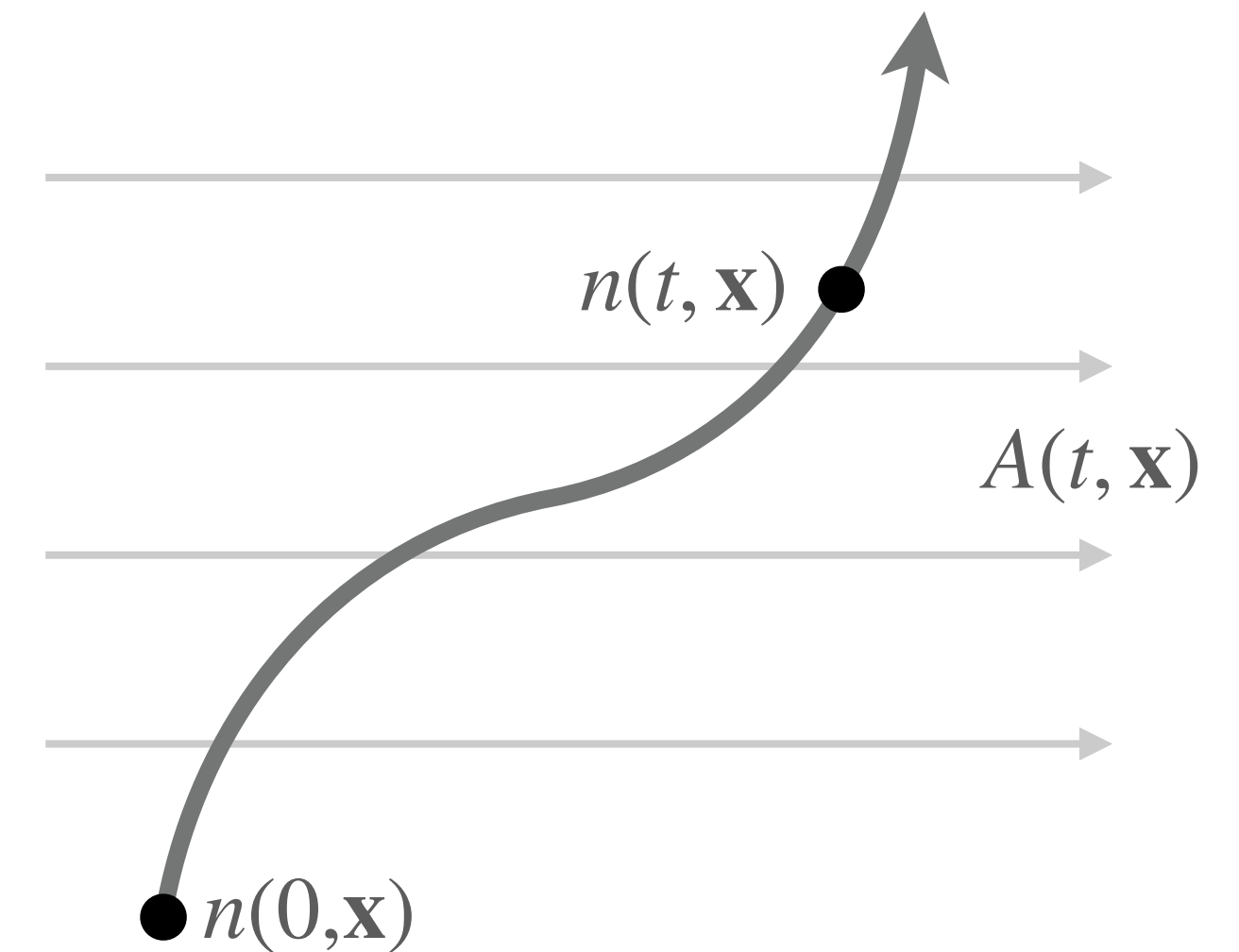
- We can introduce background sources: *gauge field* A_μ coupled to J^μ .
The theory must be invariant under background gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

- Hydrodynamic systems are characterised by their **constitutive relations**

$$\mathbf{J}[n, A_\mu] = -D(n) \nabla n + \sigma(n) \mathbf{E} + \dots \text{higher derivatives} \dots$$

$$\mathbf{E} = \partial A_t - \partial_t \mathbf{A}$$



*For a relativistic theory, one should work with the proper density, i.e. $J^t = (1 - v^2)^{-1/2} n$.

CONSTITUTIVE RELATIONS

- Constitutive relations are constrained by *symmetries*, and phenomenological constraints such as the *local second law of thermodynamics*, *existence of thermal equilibrium*, and *Onsager reciprocity relations*.¹
- For example, the local second law of thermodynamics postulates an entropy current S^μ , i.e.

$$\text{Equation of state: } S^t = s(\epsilon, n) + \dots$$

$$\text{First law: } d\epsilon = Tds + \mu dn$$

$$\text{Heat flux: } T \mathbf{S} = -\mu(\epsilon, n) \mathbf{J} + \dots$$

Isothermal assumption: T constant and $\partial_t \epsilon = \mathbf{J} \cdot \mathbf{E}$.*

- Entropy must be locally produced

$$\partial_\mu S^\mu = -\frac{1}{T} \mathbf{J} \cdot (\nabla \mu - \mathbf{E}) + \dots = \frac{\sigma}{T} \left(\frac{\chi D}{\sigma} \nabla \mu - \mathbf{E} \right) \cdot (\nabla \mu - \mathbf{E}) + \dots \geq 0 \quad \chi = \frac{\partial n}{\partial \mu}$$

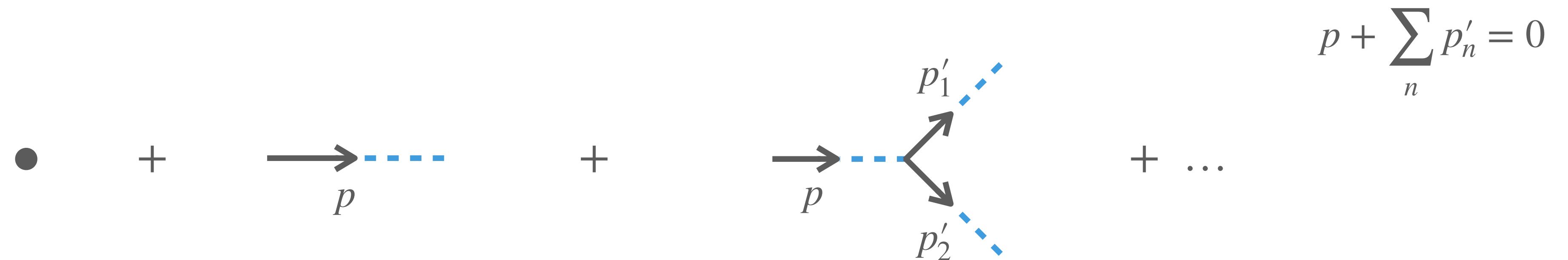
$$\Rightarrow D = \frac{\sigma}{\chi}, \quad \sigma \geq 0$$

¹Review by [Kovtun[1205.5040]] || *To relax this assumption, we need to account for a dynamical energy conservation equation in hydrodynamics.¹⁵

RESPONSE FUNCTIONS

- The classical solutions of hydrodynamic equations in Fourier-space $p = (\omega, \mathbf{k})$ can be expressed in terms of retarded correlation functions¹ or response functions as*

$$J_{\text{onshell}}(p)[A] = J_0 + \int_{p'_1} G_{JJ}^R(p; p'_1) \delta A(p'_1) + \frac{1}{2} \int_{p'_1, p'_2} G_{JJJ}^R(p; p'_1, p'_2) \delta A(p'_1) \delta A(p'_2) + \dots$$



$$G_{JJJ\dots}^R(p; p'_1, p'_2) = \left(\frac{\delta}{\delta A(p'_1)} \frac{\delta}{\delta A(p'_2)} \dots \right) J_{\text{onshell}}(p)[A] \Big|_{\delta A=0}$$

RESPONSE FUNCTIONS

- For example, 2- and 3-point correlation functions of density are given as

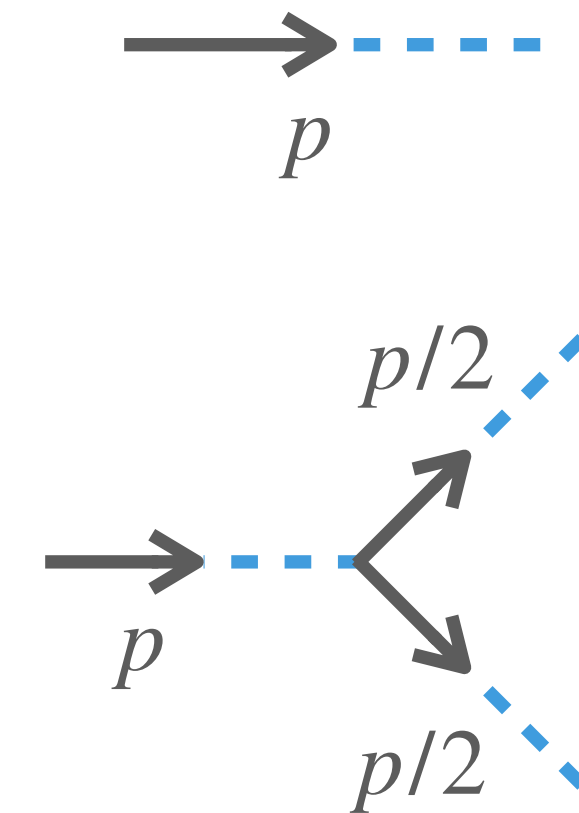
$$G_{J^t J^t}^R(p) \equiv G_{J^t J^t}^R(p; -p) = \frac{ik^2 \sigma}{\omega + iD \mathbf{k}^2} + \dots$$

$$G_{J^t J^t J^t}^R(p; -p/2, -p/2) = \frac{ik^4 \sigma^2/4 (\mathbf{k}^2 D' - (\mathbf{k}^2 - 2i\omega/D) \sigma')}{(\omega + iD \mathbf{k}^2) (\omega + iD \mathbf{k}^2/2)^2} + \dots$$

$$\chi = \frac{\partial n}{\partial \mu}, \quad D = \frac{\sigma}{\chi}, \quad D' = \frac{\partial D}{\partial n}, \quad \sigma' = \frac{1}{\chi} \frac{\partial \sigma}{\partial n}$$

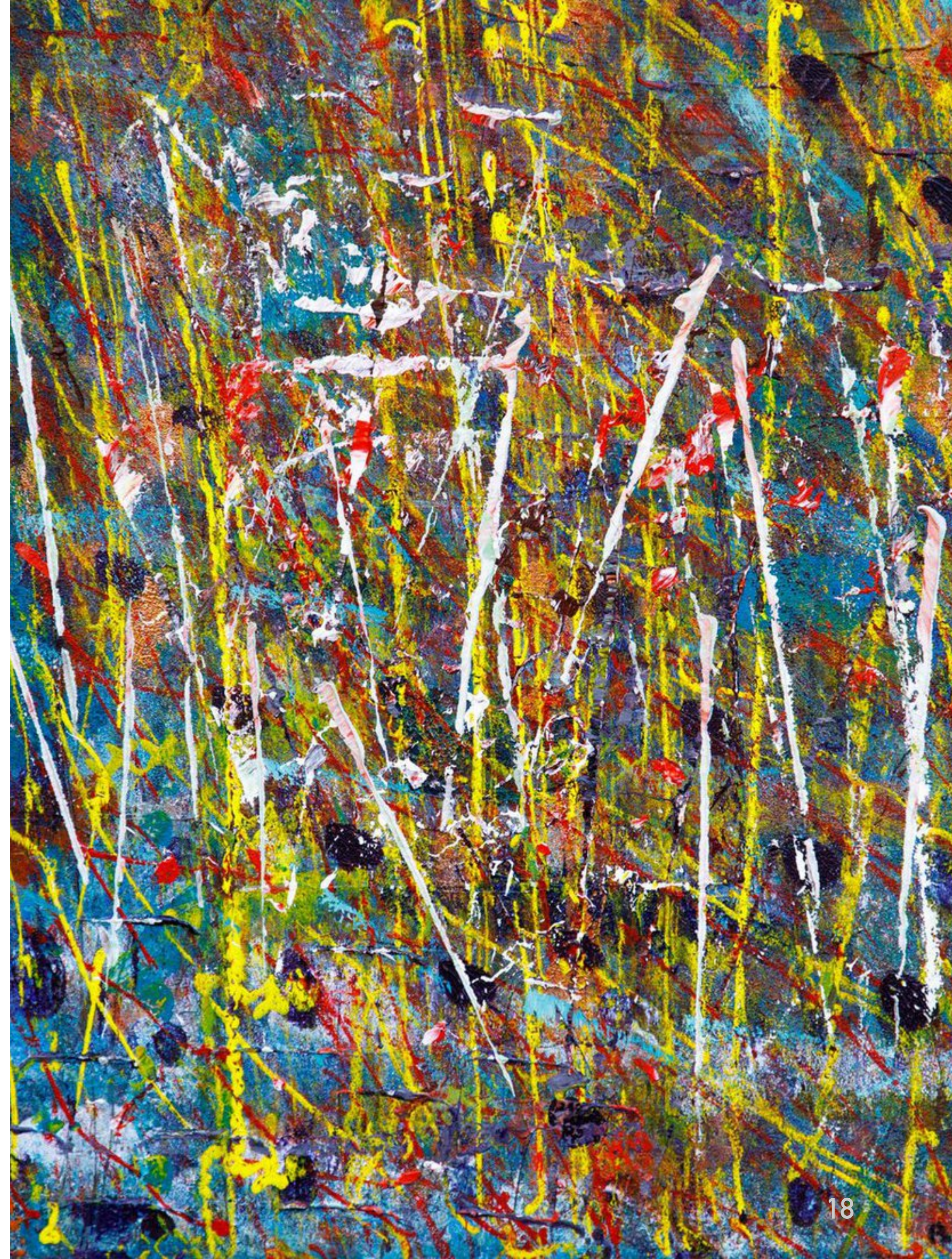
Ellipsis denote contributions coming from higher derivative terms the constitutive relations.

- Derivative corrections in classical hydrodynamics are always **analytic**, i.e. positive integer powers of ω , $\mathbf{k}^2$.



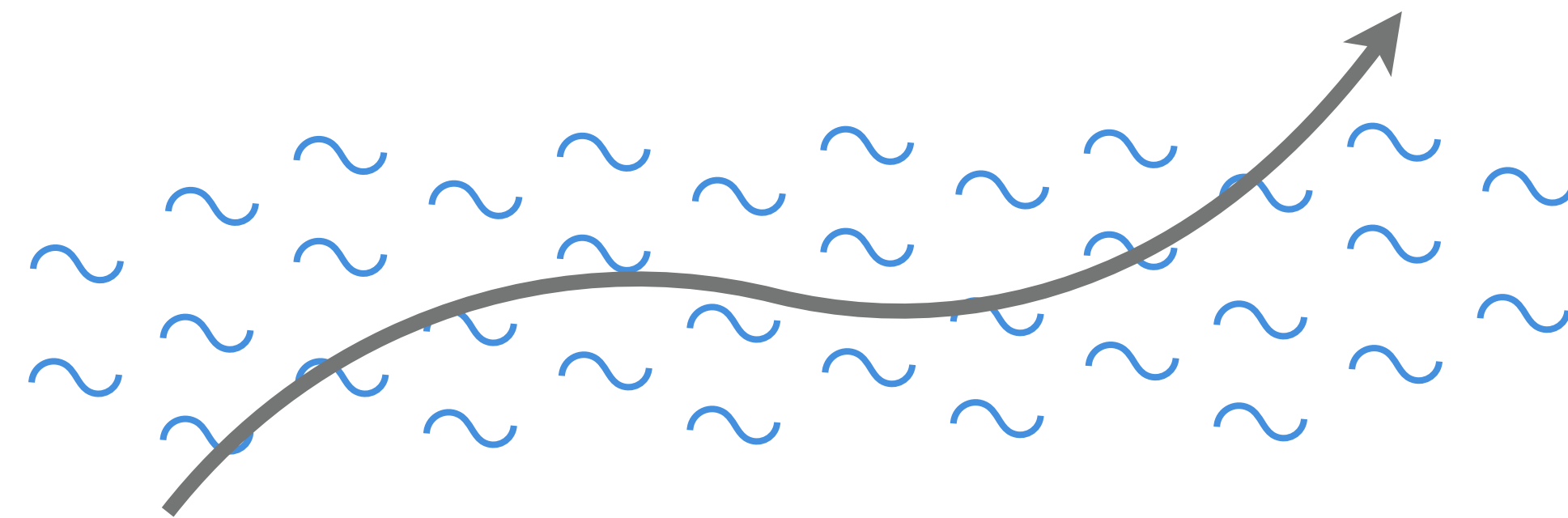
STOCHASTIC HYDRODYNAMICS

and beyond



STOCHASTIC FLUCTUATIONS

- The **classical** formulation of hydrodynamics based on conservation equations might be too simple for many purposes.
- This accounts for mutual interactions between conserved variables, but not for possible interactions with the background **thermal noise** arising due to the ignored microscopic excitations.



- These effects lead to physically observable signatures in the low-energy spectrum, known as long-time tails, that cannot be predicted by classical hydrodynamics.¹

¹[Alder, Wainwright (1970)] [De Schepper, Van Beyerren, Ernst (1974)]
[Forster, Nelson, Stephen (1977)]

STOCHASTIC HYDRODYNAMICS (AGAIN, DIFFUSION)

- Hydrodynamic observables are random due to thermal noise

$$\langle J J \dots \rangle \neq \langle J \rangle \langle J \rangle \dots$$

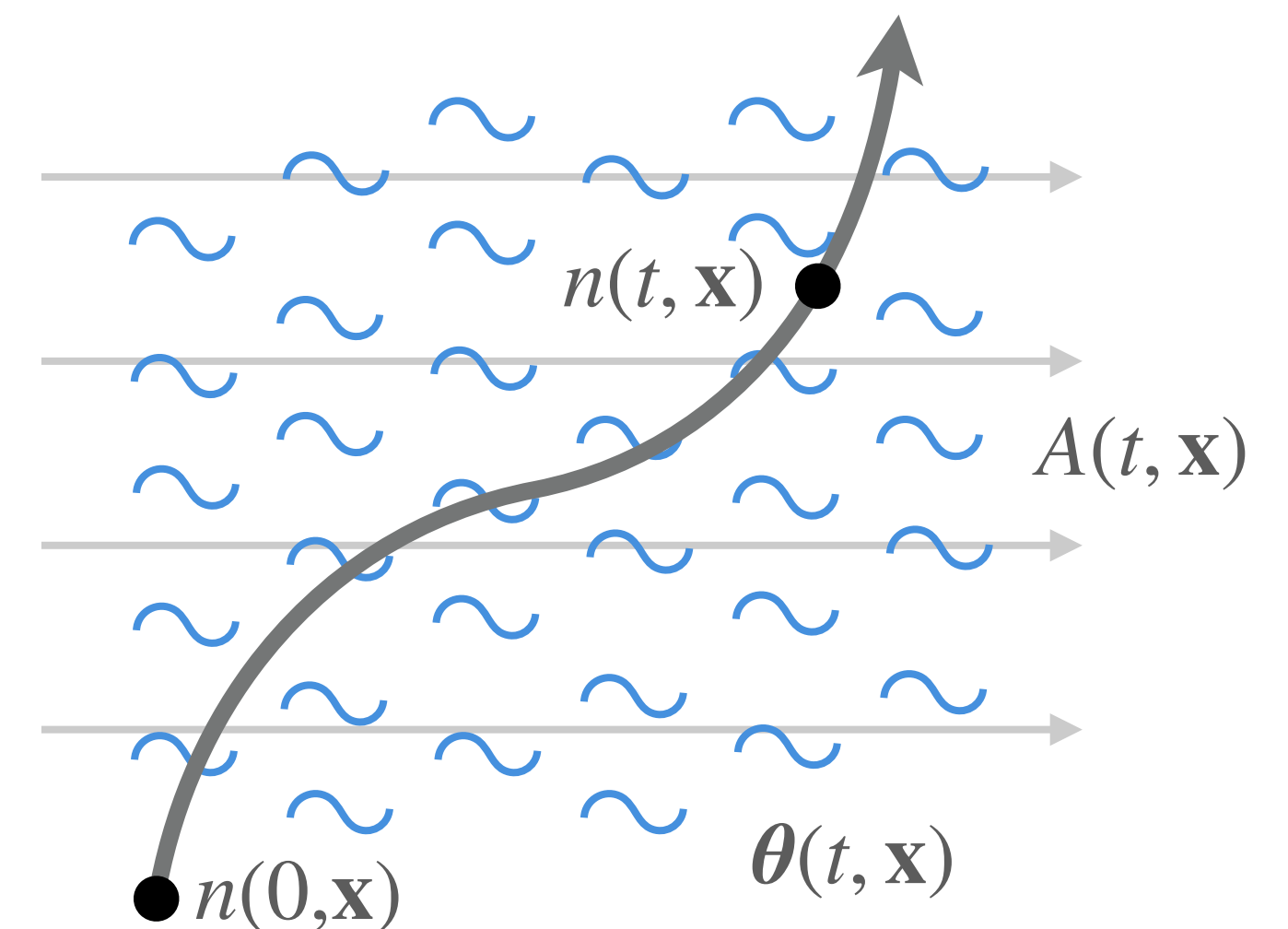
- We introduce random thermal noise¹

$$\mathbf{J} = -\sigma (\nabla \mu - \mathbf{E}) + \boldsymbol{\theta}$$

$$\langle J J \dots \rangle[A] = \int \mathcal{D}\boldsymbol{\theta} J[A, \boldsymbol{\theta}] J[A, \boldsymbol{\theta}] \dots \mathcal{P}[\boldsymbol{\theta}]$$

drawn from some probability distribution $\mathcal{P}[\boldsymbol{\theta}]$.

- First approximation: Gaussian noise $\ln \mathcal{P}[\boldsymbol{\theta}] = - \int_x C \boldsymbol{\theta}^2$.
More generally, can also have coloured or multiplicative noise.



¹[Martin, Siggia, Rose (1973)]

TIME-ORDERED CORRELATION FUNCTIONS

- Stochastic hydrodynamics can be used to compute a more general class of correlation functions*

$$G_{rr...aa...}(p_1, p_2, \dots; p'_1, p'_2, \dots) = \underbrace{\left(\frac{\delta}{\delta A(p'_1)} \frac{\delta}{\delta A(p'_2)} \dots \right)}_{\#a} \underbrace{\left(\langle J(p_1) J(p_2) \dots \rangle [A] - \text{disconnected} \right)}_{\#r} \Big|_{\delta A=0}$$

Retarded: $G_{JJ...}^R = G_{ra...}$, symmetric: $G_{JJ...}^S = G_{rr...}$, and things in between.

- Can be formally defined using a generating functional

$$W[A, A_a] = \ln \int \mathcal{D}\theta \exp \left(i \int_x J[A, \theta] A_a \right) \mathcal{P}[\theta]$$

$$G_{rr...aa...}(p_1, p_2, \dots; p'_1, p'_2, \dots) = \underbrace{\left(\frac{\delta}{\delta A(p'_1)} \frac{\delta}{\delta A(p'_2)} \dots \right)}_{\#a} \underbrace{\left(\frac{-i\delta}{\delta A_a(p_1)} \frac{-i\delta}{\delta A_a(p_2)} \dots \right)}_{\#r} W[A, A_a] \Big|_{\delta A, A_a=0}$$

*There are also “out-of-time-ordered correlation functions” (OTOCs) that cannot be accessed using stochastic hydrodynamics.

FLUCTUATION-DISSIPATION THEOREMS

- For an arbitrary non-equilibrium system, $\mathcal{P}[\theta]$ can be arbitrary.
- For thermal systems, $\mathcal{P}[\theta]$ must be such that n-point *fluctuation-dissipation theorems* (FDTs) are satisfied

$$G_{rr}(p_1, p_2) = \frac{T}{i\omega} \left(G_{ra}(p_1; p_2) - G_{ra}^*(p_1; p_2) \right)$$

$$G_{rra}(p_1, p_2; p_3) = -\frac{T}{i\omega_2} \left(G_{raa}(p_1; p_2, p_3) - G_{raa}^*(p_3; p_2, p_1) \right) + (1 \leftrightarrow 2)$$

$$G_{rrr}(p_1, p_2, p_3) = -\frac{T^2}{\omega_2 \omega_3} \left(G_{raa}(p_1; p_2, p_3) + G_{raa}^*(p_1; p_2, p_3) \right) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)$$

and analogously for higher-point functions.¹

- All non-retarded correlators G_{rr}, G_{rra}, G_{rrr} are fixed in terms of the retarded correlators G_{ra}, G_{raa} for $n = 2, 3$. This does not hold for $n \geq 4$. FDTs still give constraints but not that strong.

¹[Wang, Heinz [hep-th/9809016]]

FLUCTUATION-DISSIPATION THEOREMS

- As a consequence of FDTs, the noise distribution is completely fixed upto cubic order in fluctuations.
For diffusion model, we find

$$\ln \mathcal{P}[\boldsymbol{\theta}] = - \int_x \frac{1}{4T\sigma} \boldsymbol{\theta}^2 + \text{quartic fluctuations} \quad \sigma \geq 0$$

- At quartic and higher levels, we can get new *stochastic transport coefficients* in the noise distribution.¹
For example

$$\ln \mathcal{P}[\boldsymbol{\theta}] = - \int_x \frac{1}{4T\sigma} \boldsymbol{\theta}^2 - \frac{\vartheta_1}{(2T\sigma)^2} ((\mathbf{V} \cdot \boldsymbol{\theta})^2 - \mathbf{V}^2 \boldsymbol{\theta}^2) + \frac{T^2 \vartheta_2}{(2T\sigma)^4} \boldsymbol{\theta}^2 (\boldsymbol{\theta} - 2\sigma \mathbf{V})^2 + \text{quintic fluctuations}$$
$$\mathbf{V} = \nabla \mu - \mathbf{E}$$
$$\vartheta_1, \vartheta_2 \geq 0$$

These are left unfixed by FDTs.

- Stochastic transport coefficients characterise the *non-universality of hydrodynamics*, i.e truly microscopic information of the system that cannot be parametrised by derivative corrections to constitutive relations.

¹[AJ, Kovtun [2009.01356]]

LONG-TIME TAILS

- At leading order in stochastic fluctuations, the 2-point function modifies to¹



$$G_{J_t J_t}^R(p) = \frac{i\mathbf{k}^2}{\omega + iD\mathbf{k}^2} \left(\sigma + \frac{\omega\mathbf{k}^2}{\omega + iD\mathbf{k}^2} \frac{T\chi^2 D'^2}{32\pi D} \sqrt{\mathbf{k}^2 - \frac{2i\omega}{D}} + \dots \right)$$

- In real time, a state $n(0, \mathbf{k})$ at $t = 0$ evolves into²

$$n(t, \mathbf{k}) = \frac{n(0, \mathbf{k})}{i\chi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_{J_t J_t}^R(p) \frac{e^{-i\omega t}}{\omega - i\epsilon} = n(0, \mathbf{k}) \left(e^{-D\mathbf{k}^2 t} + \frac{\sqrt{2} T\chi D'^2}{16\pi^{3/2} D^{7/2}} \frac{1}{t^{3/2}} e^{-\frac{1}{2}D\mathbf{k}^2 t} + \dots \right)_{t \gg 1/(D\mathbf{k}^2)}$$

¹[De Schepper, Van Beyer, Ernst (1974)] [Forster, Nelson, Stephen (1977)]

[Arnold, Yaffe [hep-ph/9709449]] [Kovtun, Yaffe [hep-th/0303010]] [Chen-Lin, Delacretaz, Hartnoll [1811.12540]]

²Lecture notes by [Kovtun [1205.5040]]

STOCHASTIC CONTAMINATION

- The first contribution from stochastic transport coefficients comes from the two-loop diagrams



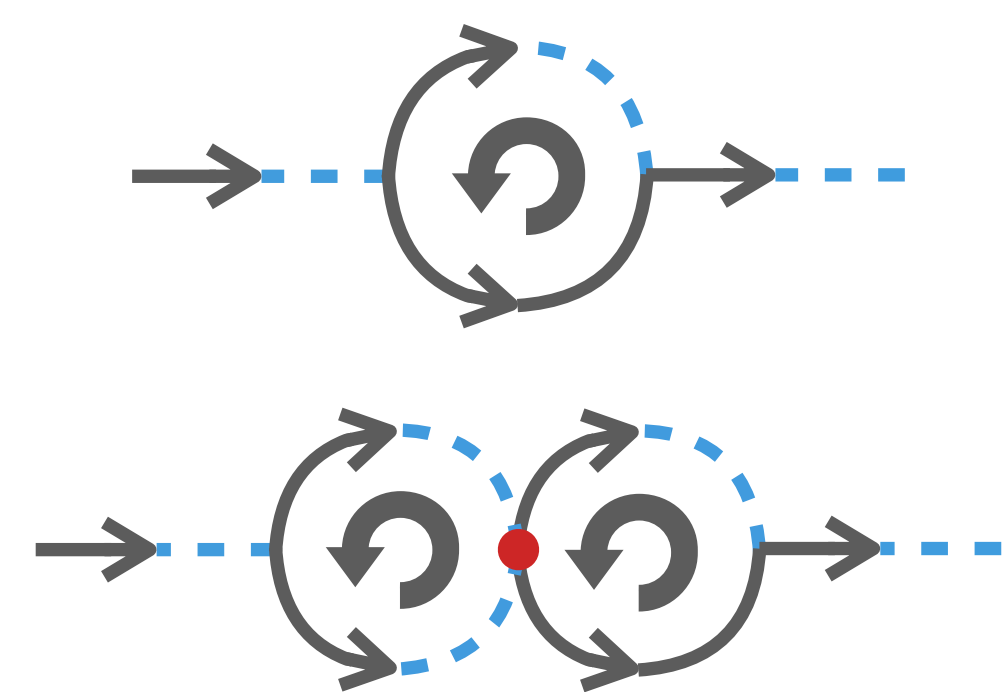
$$G_{nn}^R(\omega, \mathbf{k}) = \frac{i\mathbf{k}^2}{\omega + iD\mathbf{k}^2} \left[\sigma + \frac{\omega\mathbf{k}^2}{\omega + iD\mathbf{k}^2} \frac{T\chi^2 D'^2}{32\pi D} \sqrt{\mathbf{k}^2 - \frac{2i\omega}{D}} + \dots \right. \\ \left. - \frac{\omega\mathbf{k}^2 T D'^2}{1024\pi^2 D^2} \left(\mathbf{k}^2 - \frac{2i\omega}{D} \right) \left(\frac{1/6 \vartheta_1 \mathbf{k}^4}{\omega + iD\mathbf{k}^2} + \frac{2/3 \vartheta_1 + \vartheta_2}{D^2} (\omega + iD\mathbf{k}^2) \right) + \dots \right]$$

- First stochastic effects appear at $\mathcal{O}(\mathbf{k}^4)$, while non-universal stochastic transport coefficients appear at $\mathcal{O}(\mathbf{k}^9)$.^{*} This means that classical diffusion is reliable upto $\mathcal{O}(\mathbf{k}^3)$, while stochastic diffusion upto $\mathcal{O}(\mathbf{k}^8)$.

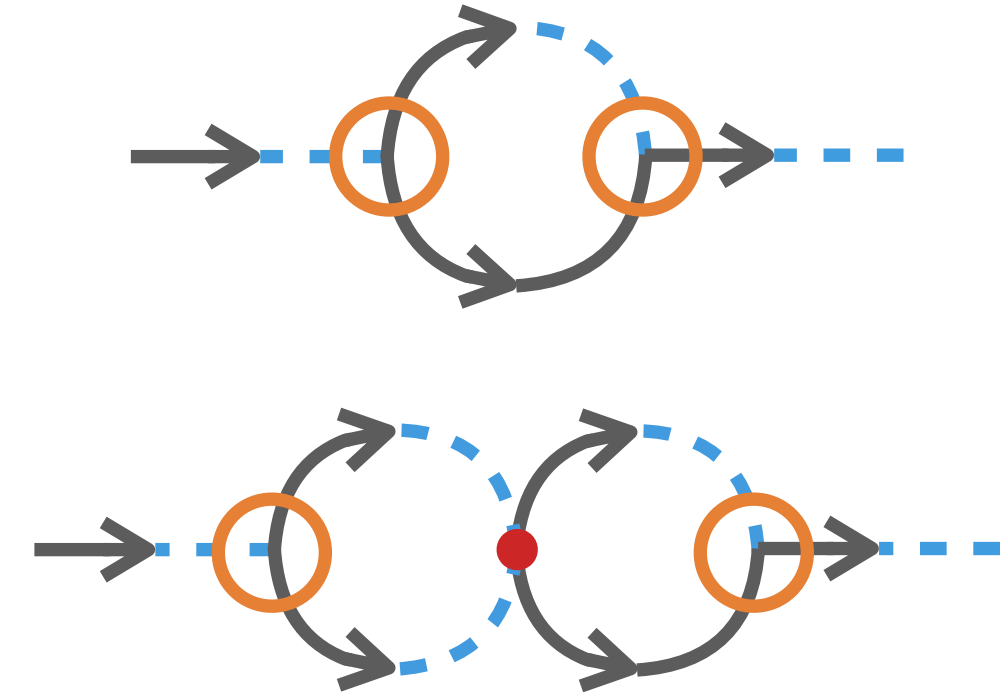
¹[AJ, Kovtun [2009.01356]] || ^{*}Compared to corrections in J^i .

SUPPRESSION OF STOCHASTIC CONTAMINATION

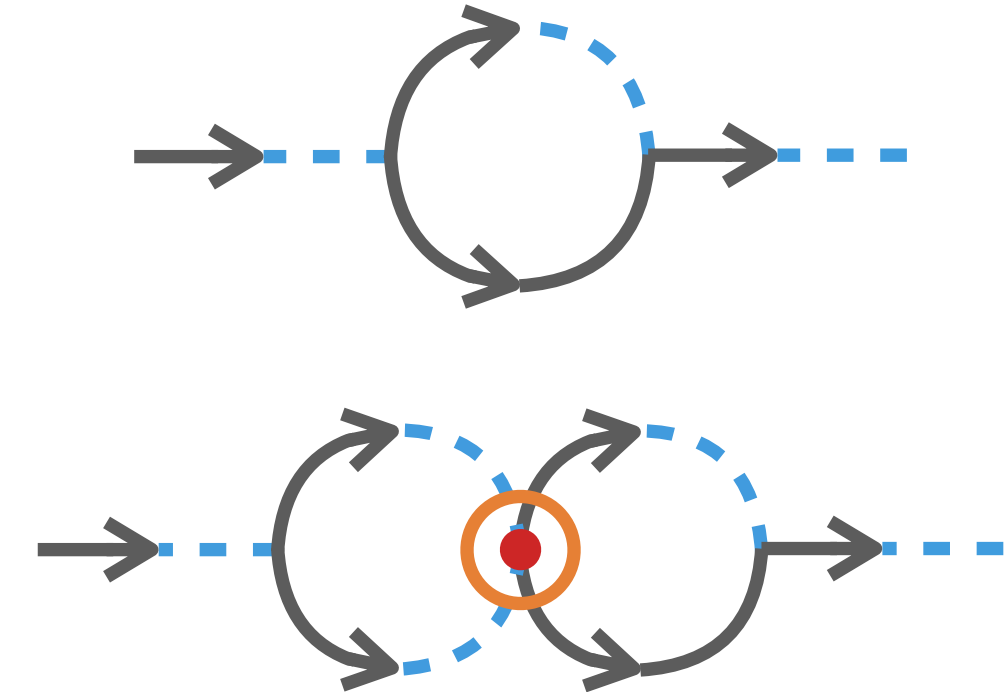
Loop Suppression



Hydrodynamic Vertices



Stochastic Vertices

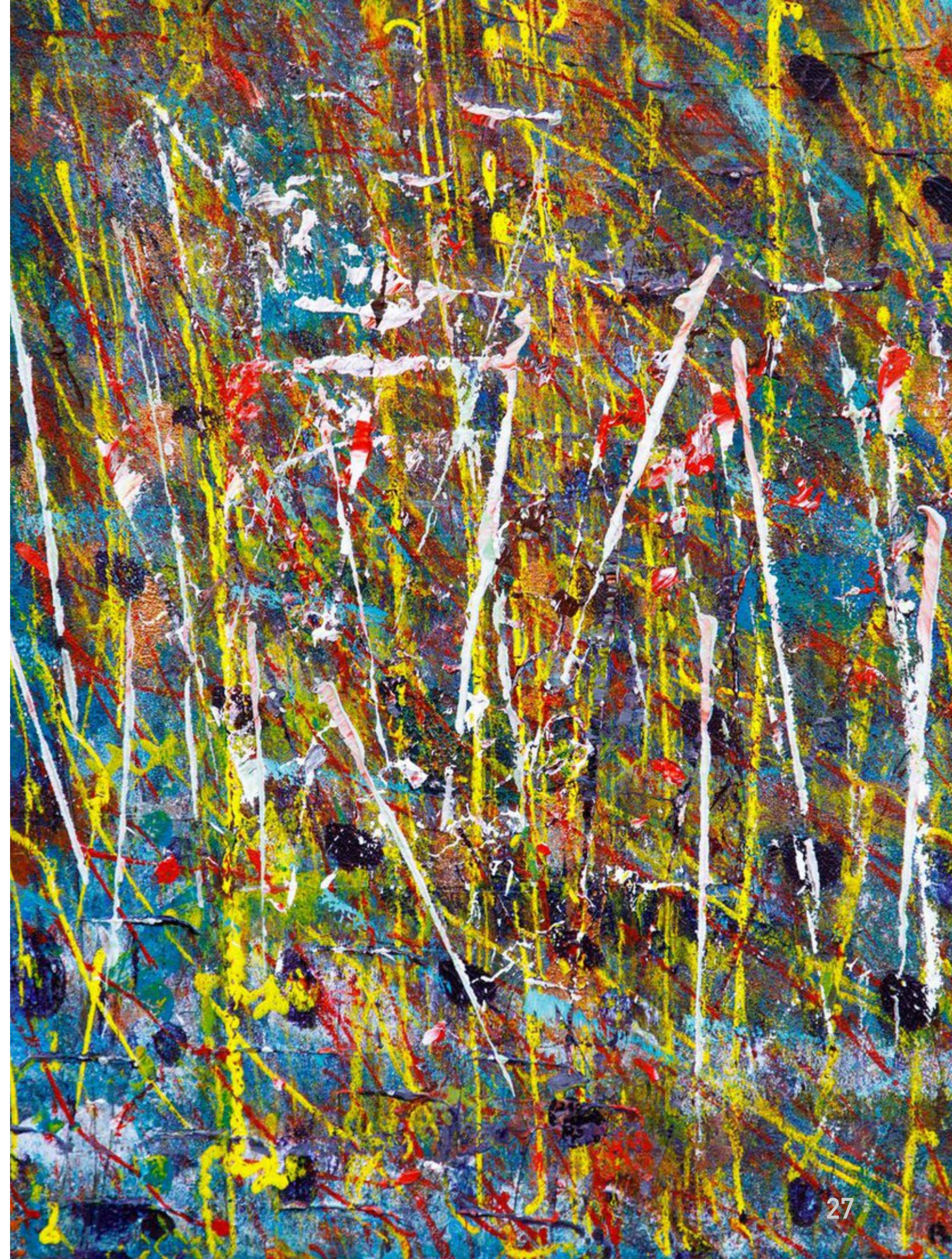


	Diffusion Model	Incompressible Hydrodynamics
Stochastic Fluctuations*	$\mathbf{k}^{d+1}$	$\mathbf{k}^{d-1}$
Non-universal Coefficients*	$\mathbf{k}^{2d+3}$	$\mathbf{k}^{2d+1}$

*Extra factors of $\log(k)$ per loop arise in $d = 2$ spatial dimensions.

EFFECTIVE FIELD THEORY FOR HYDRODYNAMICS

effective action formalism





SCHWINGER-KELDysh FRAMEWORK

- **Schwinger-Keldysh (SK)** framework for hydrodynamics provides a systematic prescription to introduce (non-Gaussian) stochastic noise into the hydrodynamic setup.¹
- Non-linear FDTs are built into the SK framework due to dynamical **Kubo-Martin-Schwinger symmetry**.
- The framework proposes a set of effective low-energy degrees of freedom and associated *symmetries* that can be used to construct an **effective action for hydrodynamics** from scratch.
- This can be used to systematically classify the possible **stochastic transport coefficients** and contribute their contribution to hydrodynamic observables.

¹[Crossley, Glorioso, Liu [1511.03646]] [Haehl, Loganayagam, Rangamani [1511.07809]]
[Jensen, Pinzani-Fokeeva, Yarom [1701.07436]]

EFFECTIVE ACTION

- We can manipulate the generating functional to give

$$W[A, \mathbf{A}_a] = \ln \int \mathcal{D}\boldsymbol{\theta} \exp \left(i \int_x \langle J \rangle_{\boldsymbol{\theta}} \mathbf{A}_a \right) \mathcal{P}[\boldsymbol{\theta}]$$

$$= \ln \int \mathcal{D}\boldsymbol{\theta} \mathcal{D}\varphi_a \mathcal{D}\mu \exp \left(i \int_x -\varphi_a \left(\partial_t n - \nabla \cdot (\sigma \mathbf{V} - \boldsymbol{\theta}) \right) + n \mathbf{A}_{at} - \sigma \mathbf{V} \cdot \mathbf{A}_a \right) \mathcal{P}[\boldsymbol{\theta}]$$

$$\mathbf{V} = \nabla \mu - \mathbf{E}$$

- Performing the noise integral under Gaussian approximation:¹ $W[A, \mathbf{A}_a] = \ln \int \mathcal{D}\varphi_a \mathcal{D}\mu \exp (iS[\mu, \varphi_a; A, \mathbf{A}_a])$

$$S = \int_x n \mathbf{B}_{at} + iT\sigma \mathbf{B}_a \cdot \left(\mathbf{B}_a + \frac{i}{T} \mathbf{V} \right)$$

$$+ i\vartheta_1 ((\mathbf{V} \cdot \mathbf{B}_a)^2 - \mathbf{V}^2 \mathbf{B}_a^2) + iT^2 \vartheta_2 \mathbf{B}_a^2 \left(\mathbf{B}_a + \frac{i}{T} \mathbf{V} \right)^2 + \text{quintic interactions}$$

$$\mathbf{B}_a = \partial \varphi_a + \mathbf{A}_a$$

¹[Kovtun, Moore, Romatschke [1405.3967]] [Harder, Kovtun, Ritz [1502.03076]]

EFFECTIVE ACTION

- Relabel $A = A_r$ and define $\mu = \partial_t \varphi_r + A_{rt}$, leading to

$$S = \int_x n(B_{rt}) \mathbf{B}_{at} + iT\sigma(B_{rt}) \mathbf{B}_a \cdot \left(\mathbf{B}_a + i\mathfrak{L}_\beta \mathbf{B}_r \right)$$

$$+ iT^2 \vartheta_1(B_{rt}) \left((i\mathfrak{L}_\beta \mathbf{B}_r \cdot \mathbf{B}_a)^2 - (\mathfrak{L}_\beta \mathbf{B}_r)^2 \mathbf{B}_a^2 \right) + iT^2 \vartheta_2(B_{rt}) \mathbf{B}_a^2 \left(\mathbf{B}_a + i\mathfrak{L}_\beta \mathbf{B}_r \right)^2$$

$$B_{r,a} \equiv \partial \varphi_{r,a} + A_{r,a}$$

$$\mathfrak{L}_\beta \equiv \frac{1}{T} \partial_t$$

$$\mu = B_{rt}$$

$$\mathbf{V} = \nabla \partial_t \varphi_r + \partial_t \mathbf{A}_r = \partial_t \mathbf{B}_r$$

- Schwinger-Keldysh conditions:

$$S[\varphi_r, \varphi_a = 0; A, A_a = 0] = 0$$

$$S[\varphi_r, -\varphi_a; A, -A_a] = -S^*[\varphi_r, \varphi_a; A, A_a]$$

$$\text{Im } S[\varphi_r, \varphi_a; A, A_a] \geq 0.$$

- $\mathbb{Z}_2$ Kubo-Martin-Schwinger (KMS) symmetry*

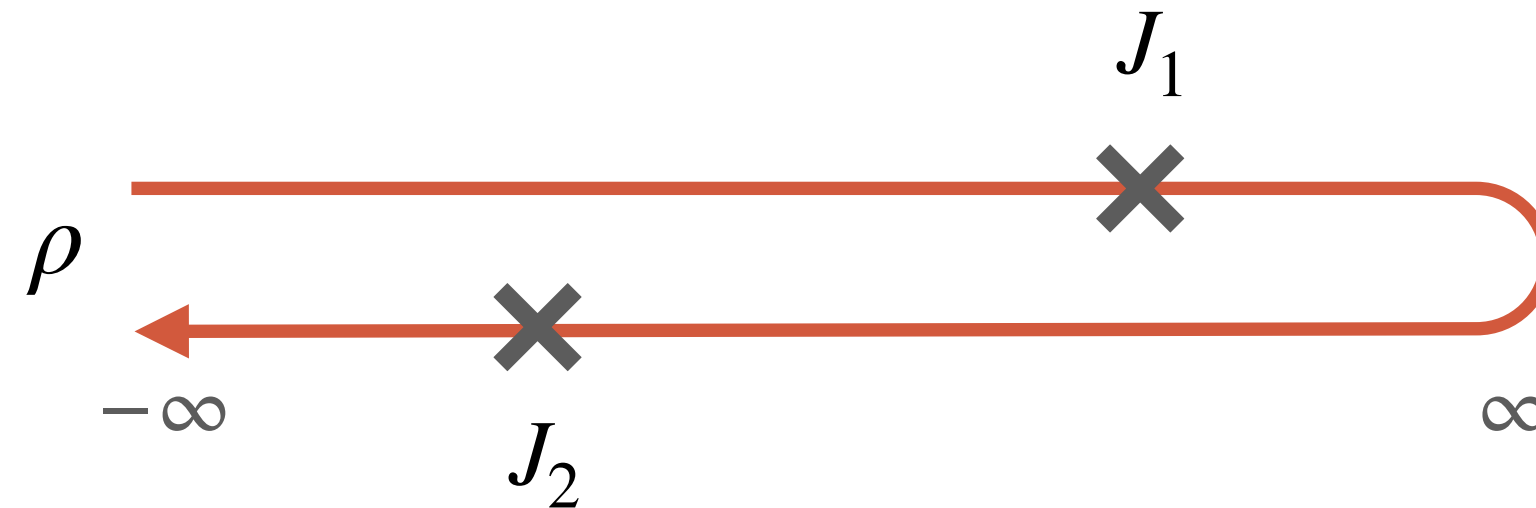
$$\varphi_r \leftrightarrow -\varphi_r \Big|_{-x}, \quad \varphi_a \leftrightarrow -\varphi_a - i\mathfrak{L}_\beta \varphi_r \Big|_{-x}$$

$$A_r \leftrightarrow A_r \Big|_{-x}, \quad A_a \leftrightarrow A_a + i\mathfrak{L}_\beta A_r \Big|_{-x}$$

*This assumes that the microscopic system realises PT symmetry. Analogous versions exist for T, CT, or CPT.

SCHWINGER-KELDysh & CLOSED-TIME CONTOUR

► Closed-time path integral¹



$$W[A_1, A_2] = \ln \text{tr}_C \left[\rho \exp \left(\frac{i}{\hbar} \int_x J_1 A_1 - J_2 A_2 \right) \right]$$

$$f_r = \frac{f_1 + f_2}{2}$$

$$W[A_r, A_a] = \ln \text{tr}_C \left[\rho \exp \left(i \int_x J_r A_a - J_a A_r \right) \right]$$

$$f_a = \frac{f_1 - f_2}{\hbar}$$

► 2-point functions

$$G_{ra}(t_1, t_2) = \frac{i}{\hbar} \Theta(t_1 - t_2) \left\langle [J(t_1), J(t_2)] \right\rangle$$

$$G_{rr}(t_1, t_2) = \frac{1}{2} \left\langle \{J(t_1), J(t_2)\} \right\rangle$$

¹[Field Theory of Non-Equilibrium Systems (Textbook), Kamenev (2011)]

SCHWINGER-KELDYSH CONDITIONS

$$W[A_r, A_a] = \ln \text{tr}_C \left[\rho \exp \left(i \int_x J_r^\mu A_{a\mu} - J_a^\mu A_{r\mu} \right) \right]$$

low energy
↓

$$= \ln \int \mathcal{D}\varphi_r \mathcal{D}\varphi_a \exp \left(iS[\varphi_r, \varphi_a; A_r, A_a] \right)$$

$$W[A_r, A_a = 0] = 0$$

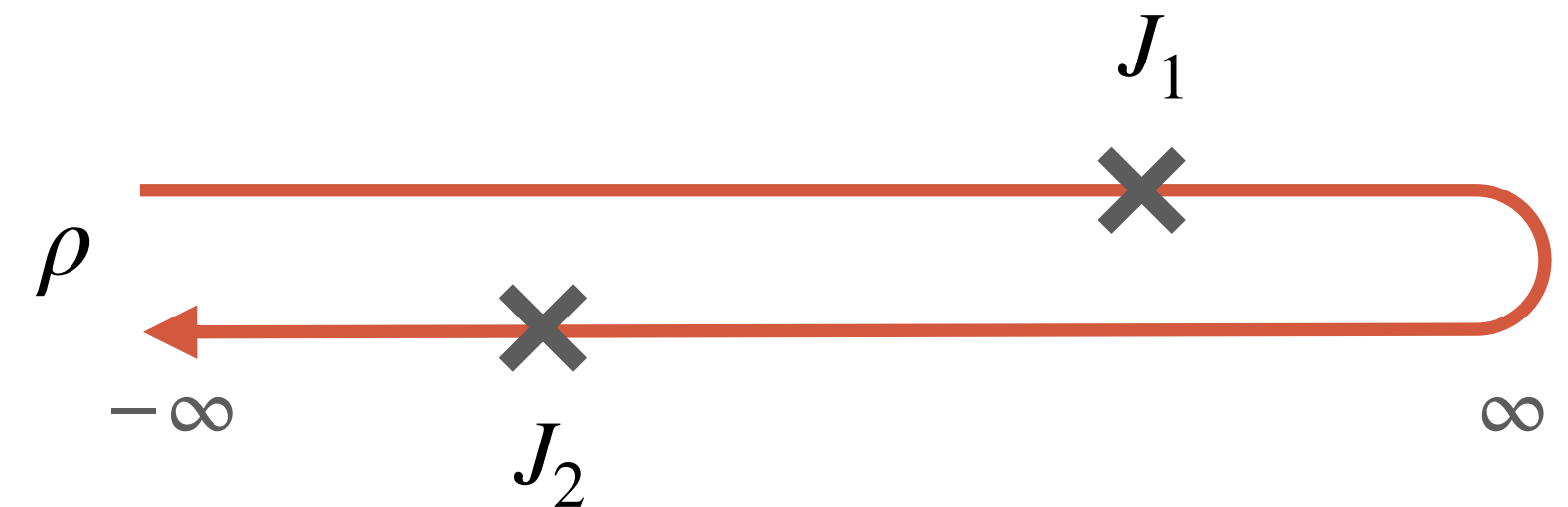
$$W[A_r, -A_a] = W^*[A_r, A_a]$$

$$\text{Re } W[A_r, A_a] \leq 0.$$

$$S[\varphi_r, \varphi_a = 0; A_r, A_a = 0] = 0$$

$$S[\varphi_r, -\varphi_a; A_r, -A_a] = -S^*[\varphi_r, \varphi_a; A_r, A_a]$$

$$\text{Im } S[\varphi_r, \varphi_a; A_r, A_a] \geq 0.$$

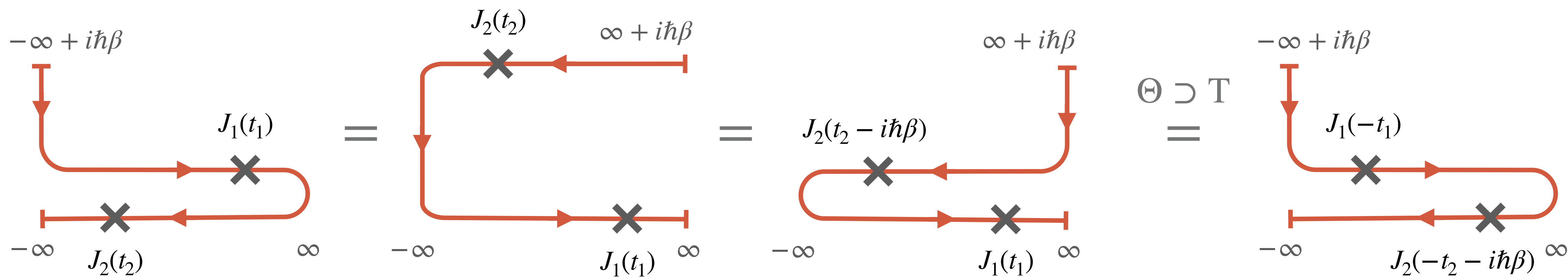


$$f_r = \frac{f_1 + f_2}{2}$$

$$f_a = \frac{f_1 - f_2}{\hbar}$$

KUBO-MARTIN-SCHWINGER (KMS) SYMMETRY

$$\rho_\beta = \exp(-\beta H) = \exp\left(-\frac{i}{\hbar}(-i\hbar\beta) H\right)$$



$$A_1 \leftrightarrow A_1 \Big|_{-x}, \quad A_2 \leftrightarrow A_2 \Big|_{-x_2 - i\hbar\beta}$$

$$\varphi_1 \leftrightarrow -\varphi_1 \Big|_{-x}, \quad \varphi_2 \leftrightarrow -\varphi_2 \Big|_{-x_2 - i\hbar\beta}$$

$\hbar \rightarrow 0$
 $\longrightarrow$

$$A_r \leftrightarrow A_r \Big|_{-x}, \quad A_a \leftrightarrow A_a + i\mathfrak{L}_\beta A_r \Big|_{-x}$$

$$\varphi_r \leftrightarrow -\varphi_r \Big|_{-x}, \quad \varphi_a \leftrightarrow -\varphi_a - i\mathfrak{L}_\beta \varphi_r \Big|_{-x}$$

SCHWINGER-KELDYSH MODEL FOR DIFFUSION

Dynamical Fields: $\varphi_{r,a}$

Background Fields: $A_{r,a}$

SK Conditions

$$f_a \rightarrow 0 \implies S \rightarrow 0$$

$$f_a \rightarrow -f_a \implies S \rightarrow -S^*$$

$$\text{Im } S \geq 0.$$

KMS Symmetry

$$f_r \leftrightarrow \Theta f_r$$

$$f_a \leftrightarrow \Theta \left(f_a + i\mathfrak{L}_\beta f_r \right)$$

Global Symmetries

$$\varphi_{r,a} \rightarrow \varphi_{r,a} - \Lambda_{r,a}$$

$$A_{r,a} \rightarrow A_{r,a} + \partial \Lambda_{r,a}$$

Diagonal Shift Symmetry

$$\varphi_r \rightarrow \varphi_r + \lambda(\mathbf{x})$$

$$B_{r,a} \equiv \partial \varphi_{r,a} + A_{r,a}$$

$$\mathfrak{L}_\beta \equiv \frac{1}{T} \partial_t$$

$$S = \int_x n(B_{rt}) \mathbf{B}_{at} + iT\sigma(B_{rt}) \mathbf{B}_a \cdot \left(\mathbf{B}_a + i\mathfrak{L}_\beta \mathbf{B}_r \right)$$

1st KMS Block

$$(G_{ra}, G_{raa}, \dots)$$

$$+ iT^2 \vartheta_1(B_{rt}) \left((i\mathfrak{L}_\beta \mathbf{B}_r \cdot \mathbf{B}_a)^2 - (\mathfrak{L}_\beta \mathbf{B}_r)^2 \mathbf{B}_a^2 \right) + iT^2 \vartheta_2(B_{rt}) \mathbf{B}_a^2 \left(\mathbf{B}_a + i\mathfrak{L}_\beta \mathbf{B}_r \right)^2$$

2nd KMS Block¹

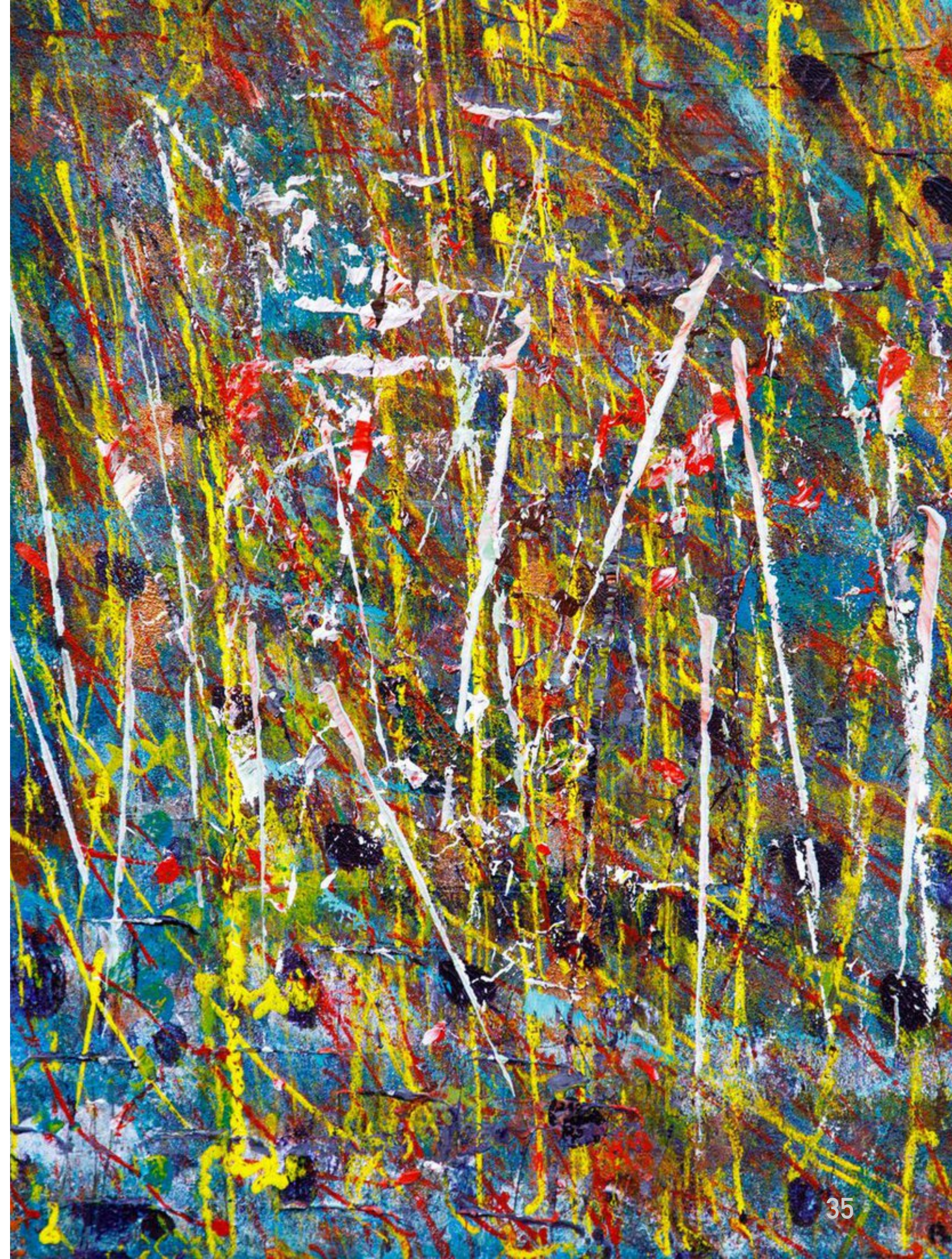
$$(G_{rraa}, G_{rraaa}, \dots)$$

n'th KMS Block $(G_{rr\dots aa\dots}^{(n,n)}, G_{rr\dots aa\dots}^{(n,n+1)} \dots)$

¹[Crossley, Glorioso, Liu [1511.03646]] [Haehl, Loganayagam, Rangamani [1511.07809]] [Jensen, Pinzani-Fokeeva, Yarom [1701.07436]]

²[AJ, Kovtun [2009.01356]]

SCHWINGER-KELDysh WITH NON-CONSERVED VARIABLES



UV-REGULATED DIFFUSION – ATTEMPT 1

- Maxwell-Cattaneo model of diffusion

$$\mathbf{V} = \nabla\mu - \mathbf{E}$$

$$\partial_t n + \nabla \cdot \mathbf{J} = 0$$

τ constant

$$\implies \partial_t n + \tau \partial_t^2 n - \sigma(n) \nabla \cdot \mathbf{V} = 0$$

$$\tau \partial_t \mathbf{J} + \mathbf{J} = -\sigma(n) \mathbf{V}$$

- MSR effective action (no background fields)¹

$$S = \int_x (n + \tau \partial_t n) \partial_t \varphi_a + iT\sigma \nabla \varphi_a \cdot \left(\nabla \varphi_a + \frac{i}{T} \mathbf{V} \right) + n A_{at}$$

- Cannot be locally coupled to A_a .
- Does not realise KMS symmetry — violate FDT for 3-point correlation functions when $\partial\sigma/\partial n \neq 0$.

¹[Abbasi, Kaminski, Tavakol [2212.11499]]

UV-REGULATED DIFFUSION – ATTEMPT 2

- BDNK model of diffusion¹

$$\mathbf{V} = \nabla \mu - \mathbf{E}$$

$$J^t = n + \tau \partial_t n$$

$$\mathbf{J} = -\sigma(n) \mathbf{V}$$

$$\implies \partial_t n + \tau \partial_t^2 n - \sigma(n) \nabla \cdot \mathbf{V} = 0$$

- MSR effective action

$$S = \int_x n \mathbf{B}_{at} - iT\chi\tau \mathbf{B}_{at} \cdot \left(\mathbf{B}_{at} + \frac{i}{T} \partial_t \mu \right) + iT\sigma \mathbf{B}_a \cdot \left(\mathbf{B}_a + \frac{i}{T} \mathbf{V} \right)$$

- Violates stability or $\text{Im } S \geq 0$.

$$G_{J^t J^t}^R(p) = \frac{(1 - i\omega\tau) i k^2 \sigma}{\omega(1 - i\omega\tau) + i D k^2}$$

$$G_{J^t J^t}^S(p) = 2T \frac{(1 + \omega^2 \tau^2) k^2 \sigma - \chi \tau D^2 k^4}{|\omega(1 - i\omega\tau) + i D k^2|^2}$$

¹[Kovtun [1907.08191]] [Bemfica, Disconzi, Noronha [1907.12695]]

UV-REGULATED DIFFUSION – ATTEMPT 3

- Maxwell-Cattaneo model of diffusion

$$\mathbf{V} = \nabla \mu - \mathbf{E}$$

$$\partial_t n + \nabla \cdot \mathbf{J} = 0$$

$$\tau \partial_t \mathbf{J} + \mathbf{J} = -\sigma(n) \mathbf{V}$$

- Schwinger-Keldysh effective action¹

$$S = \int_x n \mathbf{B}_{at} + (\mathbf{B}_a - \mathbf{J}_a) \cdot \mathbf{J} + iT\sigma \mathbf{J}_a \cdot \left(\mathbf{J}_a + \frac{i}{T} \left(\mathbf{V} + \frac{\tau}{\sigma} \partial_t \mathbf{J} \right) \right)$$

- Realises two KMS symmetries

$$\begin{array}{ccc} \mathbf{J} \leftrightarrow \mathbf{J} \Big|_{-x} & \text{(a)} & \text{(b)} \\ \mathbf{J}_a \leftrightarrow \mathbf{J}_a + \frac{i}{T} \left(\mathbf{V} + \frac{\tau}{\sigma} \partial_t \mathbf{J} \right) \Big|_{-x} & \text{or} & \mathbf{J}_a \leftrightarrow -\mathbf{J}_a - \frac{i}{T\sigma} \mathbf{J} \Big|_{-x} \end{array}$$

¹[AJ, Kovtun [2309.00511]] [Mullins, Hippert, Gavassino, Noronha [2309.00512]]

SCHWINGER-KELDYSH MODEL FOR UV-REGULATED DIFFUSION

Dynamical Fields: $\varphi_{r,a}, \mathbf{v}_{r,a}$

Background Fields: $A_{r,a}$

SK Conditions

$$\begin{aligned} f_a \rightarrow 0 &\implies S \rightarrow 0 \\ f_a \rightarrow -f_a &\implies S \rightarrow -S^* \\ \text{Im } S &\geq 0. \end{aligned}$$

KMS Symmetry

$$\begin{aligned} f_r &\leftrightarrow \Theta f_r \\ f_a &\leftrightarrow \Theta \left(f_a + i\mathfrak{L}_\beta f_r \right) \end{aligned}$$

Global Symmetries

$$\begin{aligned} \varphi_{r,a} &\rightarrow \varphi_{r,a} - \Lambda_{r,a} \\ A_{r,a} &\rightarrow A_{r,a} + \partial \Lambda_{r,a} \end{aligned}$$

Diagonal Shift Symmetry

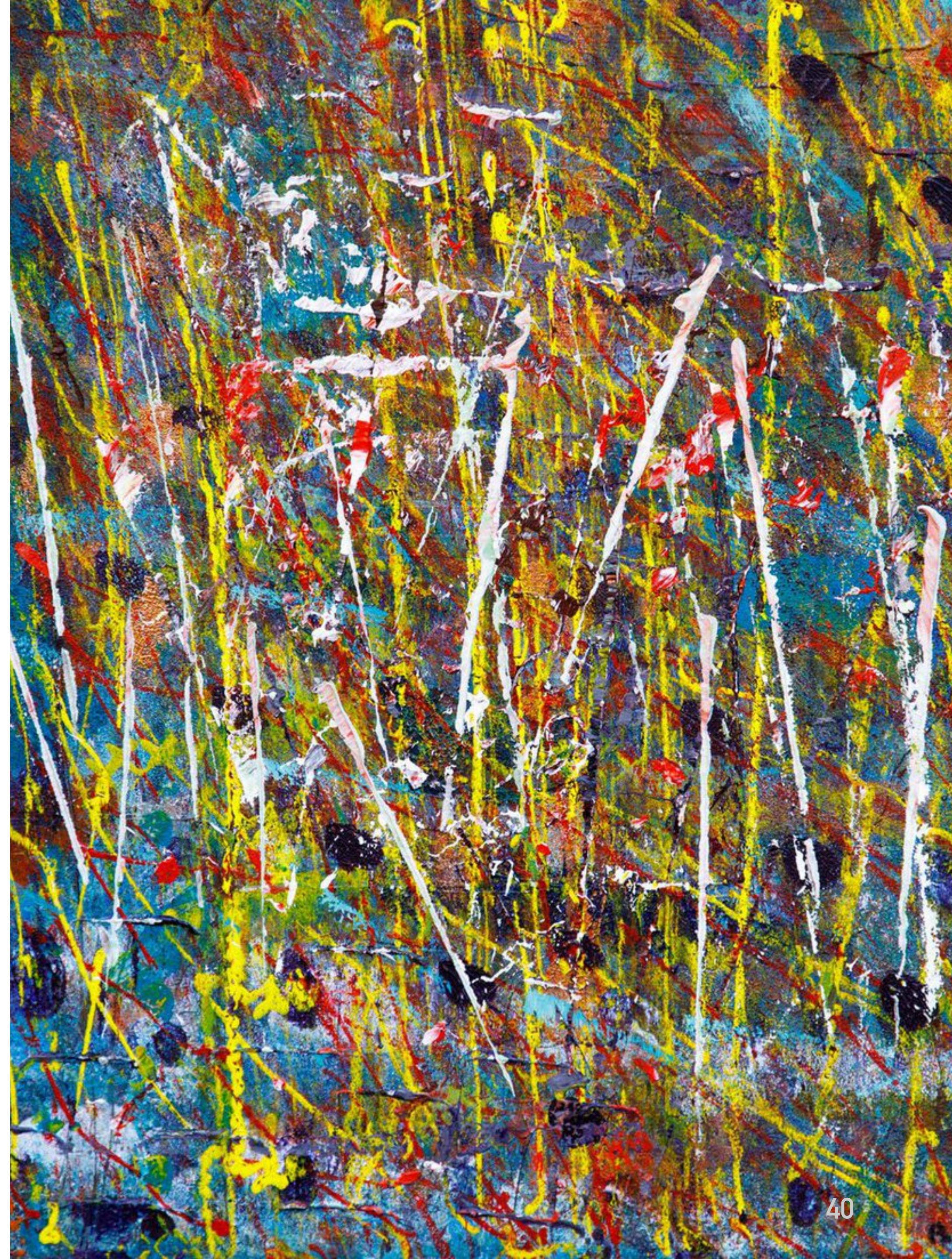
$$\begin{aligned} \varphi_r &\rightarrow \varphi_r + \lambda(\mathbf{x}) \\ \text{(b)} \quad \mathbf{v}_r &\rightarrow \mathbf{v}_r + \lambda_v(\mathbf{x}) \end{aligned}$$

$$S = \int_x n \mathbf{B}_{at} + (\mathbf{B}_a - \mathbf{J}_a) \cdot \mathbf{J} + iT\sigma \mathbf{J}_a \cdot \left(\mathbf{J}_a + \frac{i}{T} \left(\mathbf{V} + \frac{\tau}{\sigma} \partial_t \mathbf{J} \right) \right)$$

$$\text{(a)} \quad \mathbf{J} = \mathbf{v}_r, \quad \mathbf{J}_a = \mathbf{B}_a + \frac{\tau}{\sigma} \mathbf{v}_a$$

$$\text{(b)} \quad \mathbf{J} = \partial_t \mathbf{v}_r, \quad \mathbf{J}_a = \frac{1}{\sigma} \mathbf{v}_a$$

OUTLOOK





EXTENSIONS OF SCHWINGER-KELDysh

- Non-relativistic and boost-agnostic descriptions

[AJ [2008.03994]] [AJ, Armas [2010.15782]]

- Relativistic magnetohydrodynamics

[Glorioso, Son [1811.04879]] [Gralla, Iqbal [1811.04879]]

- Spontaneously broken symmetries

- $U(1)$ — superfluids

[Donos, Kailidis [2304.06008]]

[Baggioli, Bu, Ziogas [2304.14173]]

- Translations — crystals and elasticity

[Baggioli, Landry [2008.05339]]

[Baggioli, Landry, Zaccone [2101.05015]]

- Rotations — Nematic liquid crystals, etc.

- Hydrodynamics with exotic symmetries

- Dipole, multipole, and subsystem symmetries (fractons)

- Higher-form symmetries [Glorioso, Son [1811.04879]] [Gralla, Iqbal [1811.04879]]

- Higher-group and non-invertible symmetries

[Das, Iqbal, Poovuttikul [2212.09787]]



OUTLOOK

- Hydrodynamics, classical or stochastic, is **not a universal** description of long-range late-time physics.
- Late-time correlation functions can get contaminated by **stochastic transport coefficients**, characteristic of the microscopic structure of the many-body system.
- These can be captured by the general structure of thermal noise in **Schwinger-Keldysh hydrodynamics**.
- Schwinger-Keldysh hydrodynamics can be extended to **non-conserved** slowly-varying variables.
- Generalisation of Martin-Siggia-Rose prescription to **higher-point fluctuation-dissipation theorems**.

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High Energy Physics – Theory

[Submitted on 2 Sep 2020]

Non–universality of hydrodynamics

Akash Jain, Pavel Kovtun

mySpires.

Diffusion

Effective Action

Schwinger-Keldysh

Stochastic Hydrodynamics

We investigate the effects of stochastic interactions on hydrodynamic correlation functions using the Schwinger–Keldysh effective field theory. We identify new "stochastic transport coefficients" that are invisible in the classical constitutive relations, but nonetheless affect the late–time behaviour of hydrodynamic correlation functions through loop corrections. These results indicate that classical transport coefficients do not provide a universal characterisation of long–distance, late–time correlations even within the framework of fluctuating hydrodynamics.

Comments: 5 pages + Supplementary Material

Subjects: **High Energy Physics – Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); High Energy Physics – Phenomenology (hep-ph); Mathematical Physics (math-ph); Fluid Dynamics (physics.flu-dyn)

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AJ, P Kovtun [2009.01356, 2309.00511].

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PERTURBATIVE THEORY OF STOCHASTIC FLUCTUATIONS

- We can perturbatively expand the effective action in fluctuations around an equilibrium state

$$n = n_0 + \delta n, \quad \varphi_a, \quad A_{r\mu} = 0, \quad A_{a\mu} = 0$$

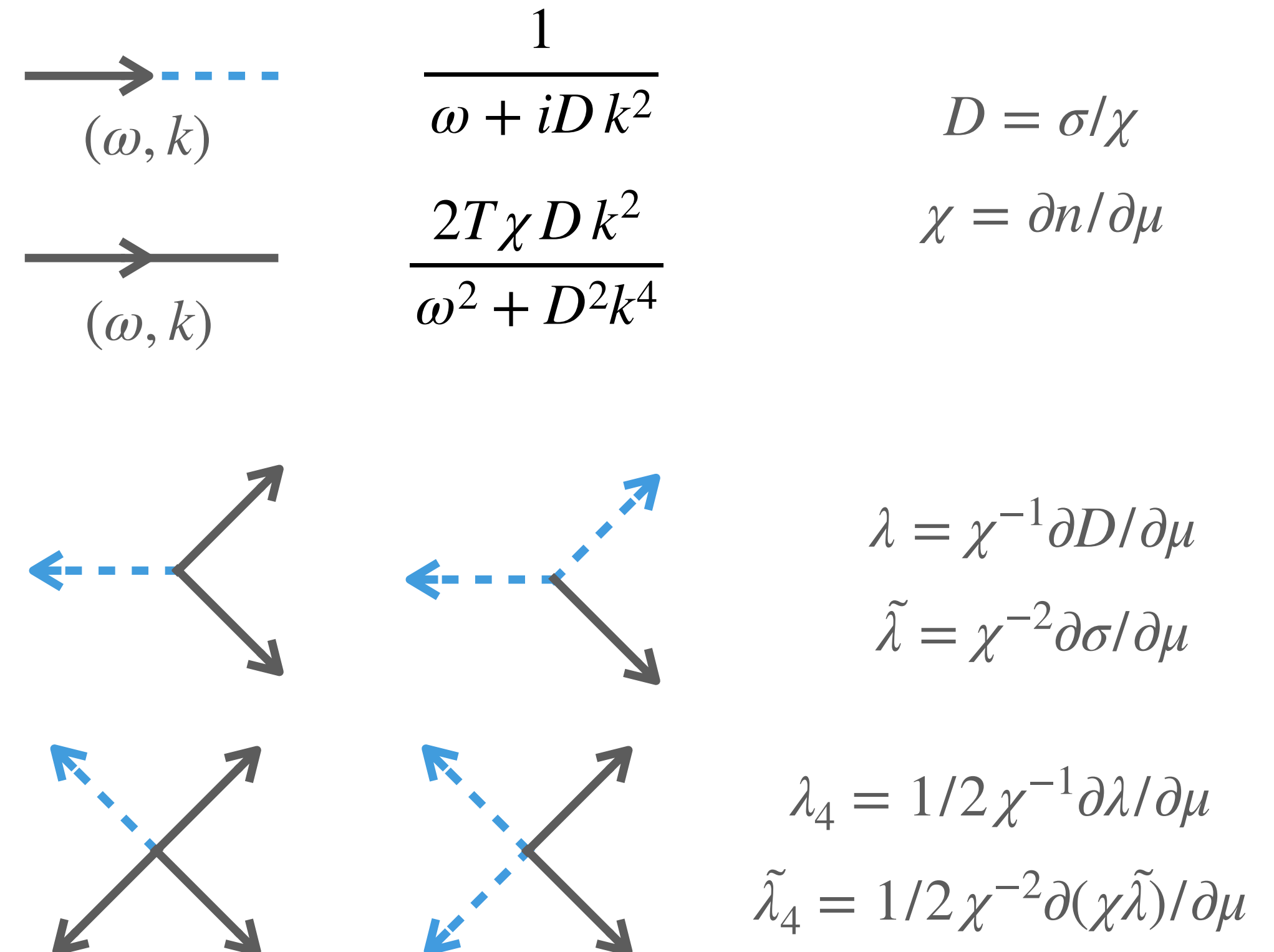
- The **classical** part of the effective action expanded to quadratic order in fields leads to the free theory

$$\mathcal{L}_1^{\text{free}} = -\varphi_a (\partial_t \delta n - D \partial^2 \delta n) + iT \sigma \partial^i \varphi_a \partial_i \varphi_a$$

- Expanding to further orders in fields, we lead to the interaction vertices¹

$$\mathcal{L}_1^{3pt} = \frac{1}{2} \lambda \delta n^2 \partial^2 \varphi_a + i\chi T \tilde{\lambda} \delta n \partial^i \varphi_a \partial_i \varphi_a$$

$$\mathcal{L}_1^{4pt} = \frac{1}{3} \lambda_4 \delta n^3 \partial^2 \varphi_a + i\chi T \tilde{\lambda}_4 \delta n^2 \partial^i \varphi_a \partial_i \varphi_a$$

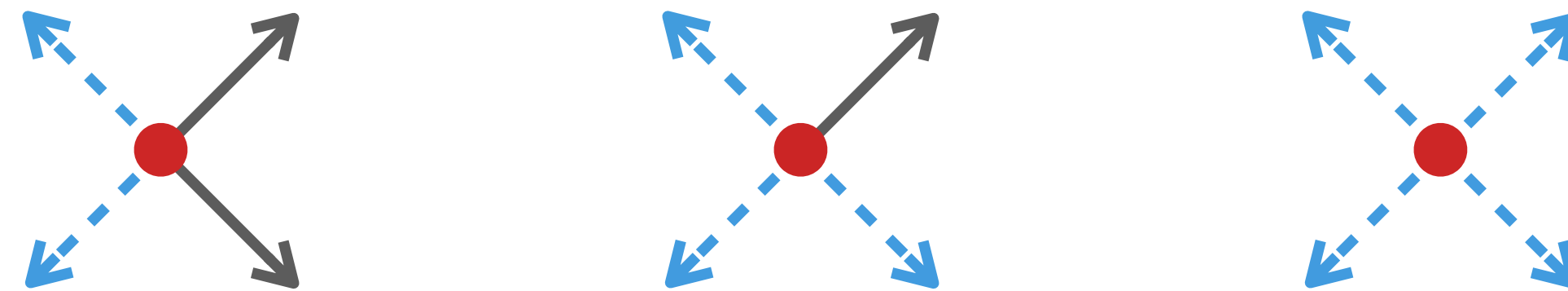


¹[Chen-Lin, Delacretaz, Hartnoll [1811.12540]]

STOCHASTIC INTERACTIONS

- Stochastic transport coefficients in the effective action lead to the quartic couplings

$$\begin{aligned}\mathcal{L}_2^{4pt} = & i \frac{\vartheta_1}{\chi^2} (\partial^i n \partial_i \varphi_a) (\partial^j n \partial_j \varphi_a) - i \frac{\vartheta_1 + \vartheta_2}{\chi^2} (\partial^i n \partial_i n) (\partial^j \varphi_a \partial_j \varphi_a) \\ & - \frac{2T \vartheta_2}{\chi} (\partial^i n \partial_i \varphi_a) (\partial^j \varphi_a \partial_j \varphi_a) + iT^2 \vartheta_2 (\partial^i \varphi_a \partial_i \varphi_a) (\partial^j \varphi_a \partial_j \varphi_a)\end{aligned}$$



- These couplings do not contribute to tree-level retarded correlation functions.