

Quantum Fluctuations in Schwinger-Keldysh Hydrodynamics



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[260x.xxxxx] AJ, Mark Mezei

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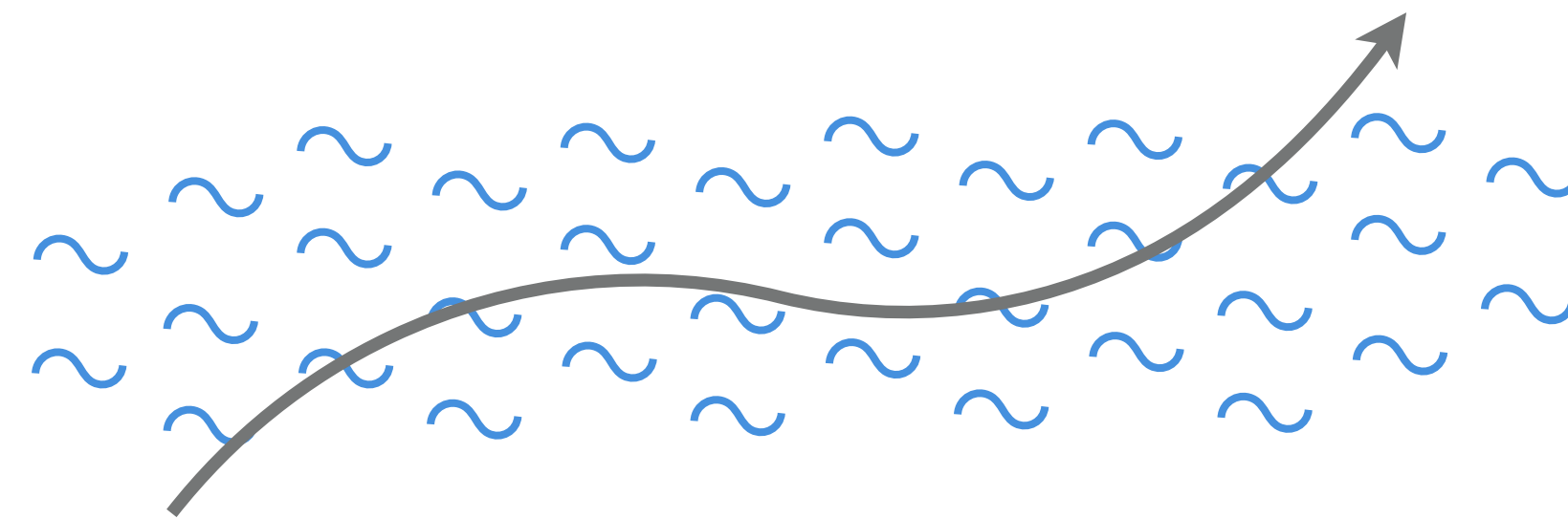
The Big Picture

- **Effective field theory (EFT)** is a powerful tool to construct low-energy effective models for complex many-body systems, without the detailed knowledge of their microscopic constituents.
- The conventional EFT framework is only suited to systems at zero temperature or systems in **thermal equilibrium** (statistical field theory).
- **Hydrodynamics** provides a universal framework to describe low-energy effective dynamics of complex many-body systems operating *close to thermal equilibrium*.
- Classical hydrodynamics is “*like*” an EFT. It involves *effective fields* for local fluid velocity, temperature, density, etc, whose dynamics is dictated by universal symmetry principles. However, it is *not* based on an **effective action principle**.

This has changed recently!

The Big Picture

- Classical hydrodynamics has limited applicability due to the presence of **fluctuations**.



- Classical hydrodynamics *includes dissipation but not fluctuations*, which are necessitated by the **fluctuation-dissipation theorem (FDT)**.
- **Schwinger-Keldysh (SK) hydrodynamics** is an EFT extension of classical hydrodynamics, which systematically incorporates the effects of fluctuations.

[Kovtun, Moore, Romatschke 2014]

[Harder, Kovtun, Ritz 2015]

[Crossley, Glorioso, Liu 2015]

[Haehl, Loganayagam, Rangamani 2015]

[Jensen, Pinzani-Fokeeva, Yarom 2017]

Fluctuation-Dissipation Theorem

$$G^S(\omega, \mathbf{k}) = \hbar \coth\left(\frac{1}{2}\hbar\beta\omega\right) \text{Im } G^R(\omega, \mathbf{k})$$

$$\frac{1}{2} \left\langle \{ \mathcal{O}(t_1), \mathcal{O}'(t_2) \} \right\rangle$$

Fluctuation

$$\frac{i}{\hbar} \Theta(t_1 - t_2) \left\langle [\mathcal{O}(t_1), \mathcal{O}'(t_2)] \right\rangle$$

Dissipation

$$\frac{2}{\beta\omega} \left(1 + \frac{\hbar^2\beta^2\omega^2}{12} + \mathcal{O}(\hbar^4\beta^4\omega^4) \right)$$

- Fluctuations can **backreact** on hydrodynamics via interactions.
- **Classical hydrodynamics** has no fluctuations!
- **Stochastic hydrodynamics** contains fluctuations at leading order in $\hbar$.

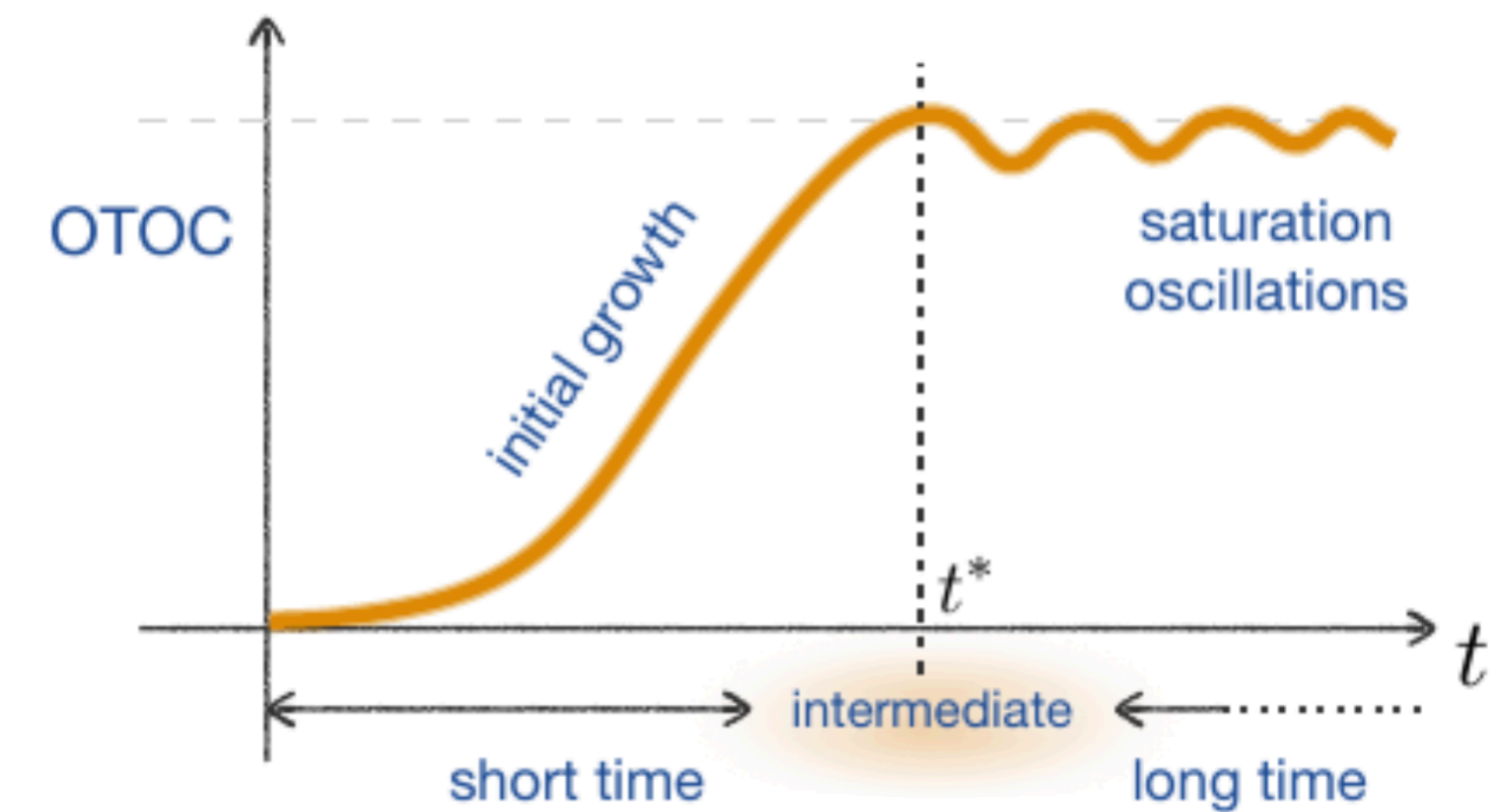
$$\omega \ll \frac{1}{t_{\text{MF}}} \ll \frac{1}{\hbar\beta}$$

Out-of-Time-Ordered Correlators (OTOCs)

- OTOCs serve as a probe of **chaos in quantum systems**

$$\left\langle [W(t), V(0)]^2 \right\rangle$$
$$\sim \exp(\lambda_L t) \quad (\text{early time})$$

Lyapunov exponent



[García-Mata et al. 2022]

- Scrambling admits a **maximal bound** [Maldacena, Shenker, Stanford 2022]

$$\lambda_L \leq \frac{2\pi}{\hbar\beta}$$

e.g. SYK, large-c CFTs

- EFTs for (nearly) maximal chaotic systems. [Blake, Lee, Liu 2018]
[Haehl, Rozali 2018]

Hydrodynamics 101

- Hydrodynamics of a conserved scalar variable n , with **conservation equation**

$$\partial_t n + \nabla \cdot \mathbf{J} = 0$$

Given $\mathbf{J} = \mathbf{J}(n)$, this equation governs the classical evolution of n .

- The **constitutive relations** are given as

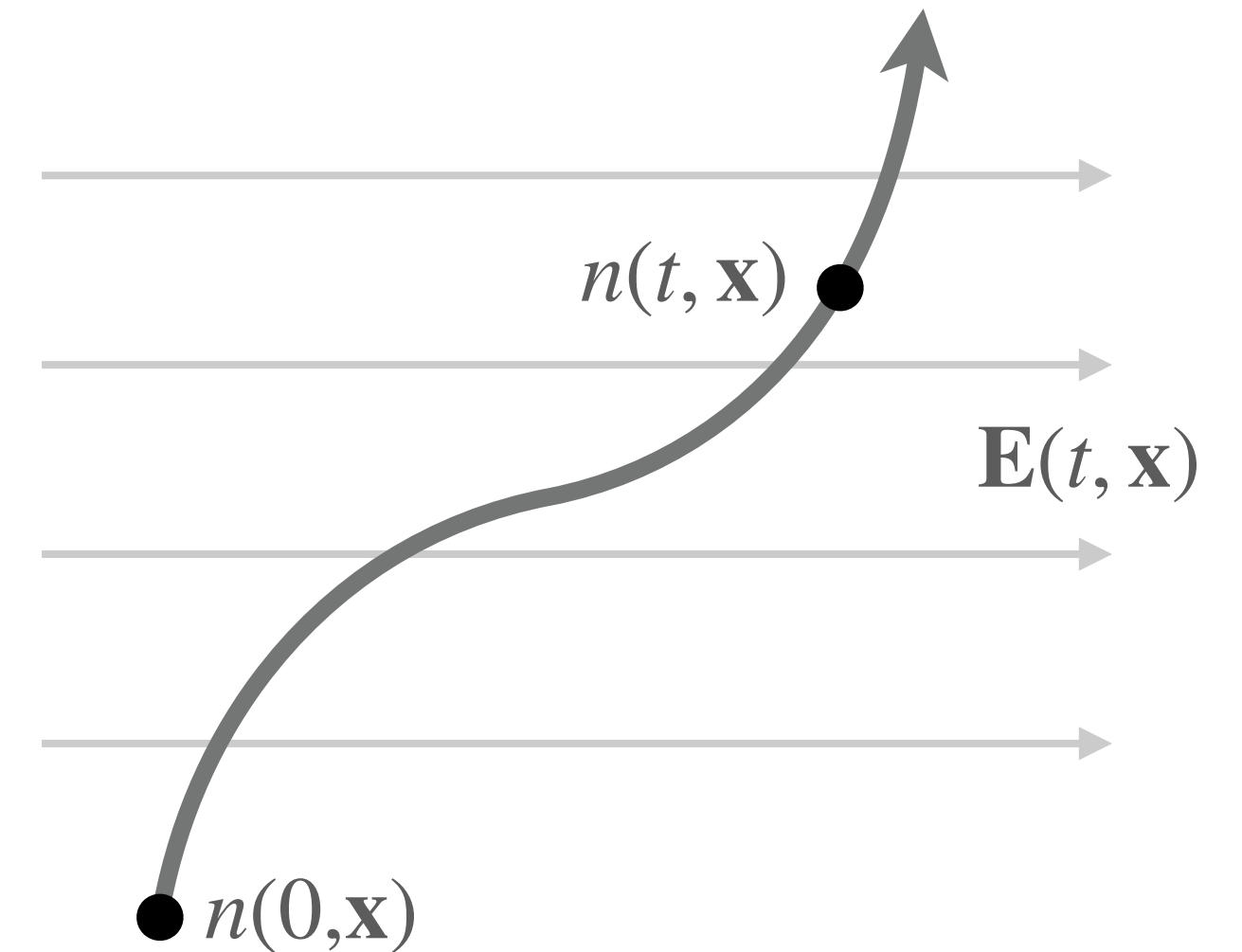
$$\Rightarrow \quad \mathbf{J} = -\sigma(n) \left(\nabla \mu(n) - \mathbf{E} \right) + \mathcal{O}(\nabla^2)$$

↓
conductivity

- Hydrodynamic prescription for **retarded correlators** [Kadanoff, Martin 1963]

$$G_{J^\mu J^\nu}^R(t_1 - t_2) = \frac{\delta}{\delta A_\nu(t_2)} J^\mu_{\text{onshell}}(t_1)$$

No fluctuations: **incompatible with FDT**



$\mathbf{E} = \nabla A_t - \partial_t \mathbf{A}$ is applied electric field

$$J^\mu = (n, \mathbf{J})$$

$$A_\mu = (A_t, \mathbf{A})$$

Schwinger-Keldysh Hydrodynamics

- All fields are doubled!
- Dynamical fields: φ_1 and φ_2 . Background fields $A_{1\mu}$ and $A_{2\mu}$.

Classical field: $\mu(n) \equiv B_{rt}$

Noise field: φ_a

Classical source: $A_\mu \equiv A_{r\mu}$

Noise source: $A_{a\mu}$

$$f_r = \frac{f_1 + f_2}{2}$$

$$f_a = \frac{f_1 - f_2}{\hbar}$$

- **Effective action** for hydrodynamics (diffusion)

$$S = \int d^{d+1}x \left(B_{a\mu} J^\mu + \mathcal{O}(f_a^2) \right)$$

engineered to satisfy **FDT**
and **unitarity**

$$\frac{\delta S}{\delta \phi_r} = 0 \implies \varphi_a = 0$$

$$\frac{\delta S}{\delta \phi_a} = 0 \implies \partial_\mu J^\mu = 0$$

(for $A_a = 0$)

$$B_{\#\mu} = \partial_\mu \varphi_\# + A_{\#\mu}$$

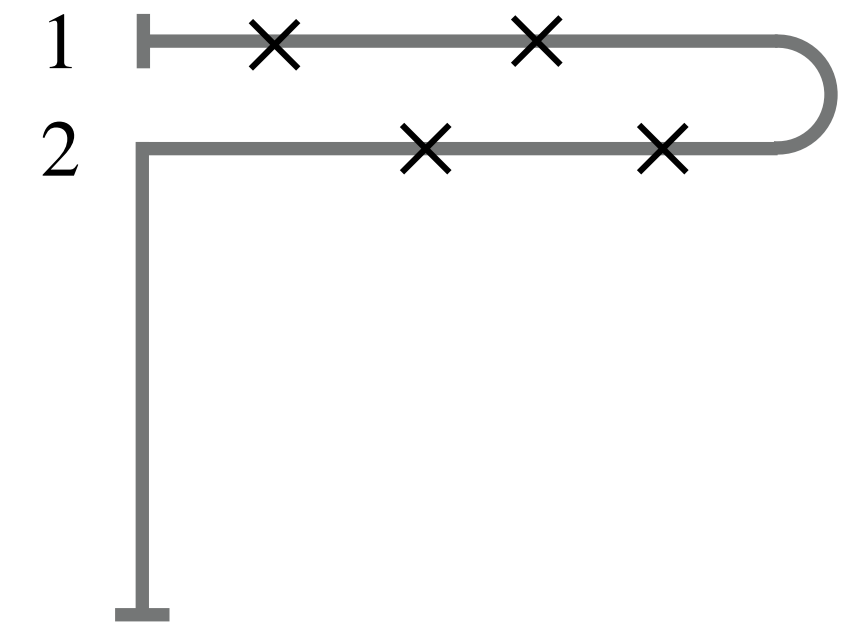
Schwinger-Keldysh Effective Action

- SK generating functional

$$\mathcal{Z}[A_{1,2}] = \int \mathcal{D}\phi_{1,2} \exp\left(i S[\phi_{1,2}; A_{1,2}]\right)$$

$$G_{JJ}^R = \frac{\delta}{\delta A_r} \frac{-i\delta}{\delta A_a} \ln \mathcal{Z}$$

$$G_{JJ}^S = \frac{-i\delta}{\delta A_a} \frac{-i\delta}{\delta A_a} \ln \mathcal{Z}$$



- Unitarity constraints

$$S_{ii} = 0$$

$$S_{ij}^* = -S_{ji}$$

$$\text{Im } S_{ij} \geq 0$$

$$F_i \equiv F[f_i]$$

$$F_{ij} \equiv F[f_i, f_j]$$

$$S_{12} = \boxed{S_1 - S_2} + \boxed{\mathcal{A}_{12}}$$

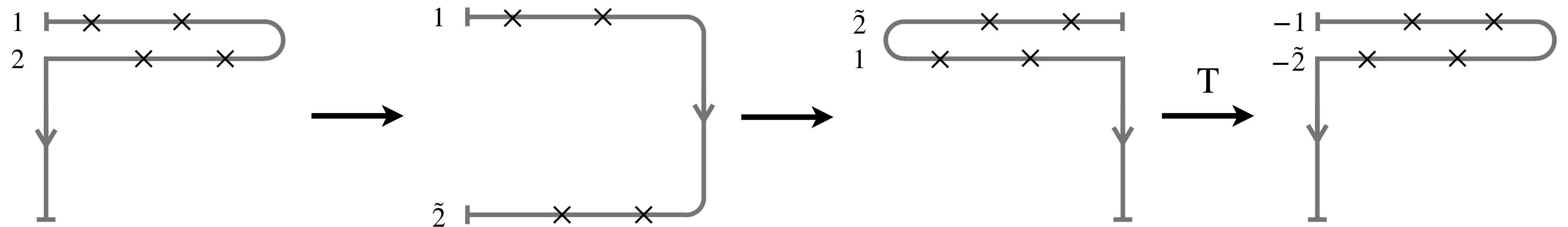
Hydrostatic

Dissipative

“Influence Functional”

KMS Symmetry

- FDT is non-linearly realised by the KMS symmetry



$$Kf_1(t) = \pm f_1(-t)$$

$$Kf_2(t) = \pm f_2(-t - i\hbar\beta)$$

$$= \pm e^{-i\hbar\beta\partial_t} f_2(-t)$$

$$K^2 = 1$$

$$\longrightarrow \mathbb{Z}_2$$

$$\tilde{i} : f_i(t - i\hbar\beta)$$

$$-i : \pm f_i(-t)$$

time-reversal eigenvalue

Schwinger-Keldysh Effective Action

S_{12}	$S_1 - S_2$	$\mathcal{A}_{12}$
Free Action	$\frac{\chi_0}{2\hbar} (B_{1t}^2 - B_{2t}^2) = \chi_0 B_{at} B_{rt}$	$-\sigma_0 \mathbf{B}_a \cdot \left(\mathcal{M} \dot{\mathbf{B}}_r - \frac{i}{\beta} \mathbf{B}_a \right)$
3-point Interactions	$\frac{\chi_1}{6\hbar} (B_{1t}^3 - B_{2t}^3) = \frac{\chi_1}{2} \left(B_{at} B_{rt}^2 + \frac{\hbar^2}{12} B_{at}^3 \right)$	$-\sigma_1 \left(B_{rt} \mathbf{B}_a - \frac{i\hbar^2\beta}{4} B_{at} \mathcal{M} \dot{\mathbf{B}}_a \right) \cdot \left(\mathcal{M} \dot{\mathbf{B}}_r - \frac{i}{\beta} \mathbf{B}_a \right)$

$$n(\mu) = \chi_0 \mu + \frac{1}{2} \chi_1 \mu^2 + \dots$$

$$\sigma(\mu) = (\sigma_0 + \sigma_1 \mu + \dots) \mathcal{M}$$

$$\mathcal{M} \equiv \frac{2}{\hbar\beta\partial_t} \tan\left(\frac{\hbar\beta\partial_t}{2}\right) = 1 + \mathcal{O}(\hbar\beta\partial_t)$$



Stochastic Interactions

Quantum Interactions

Backreaction on Retarded Correlators

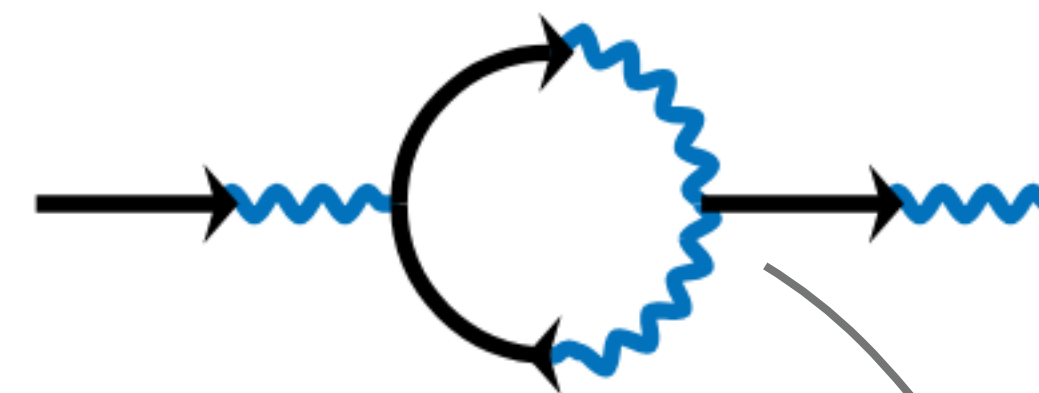
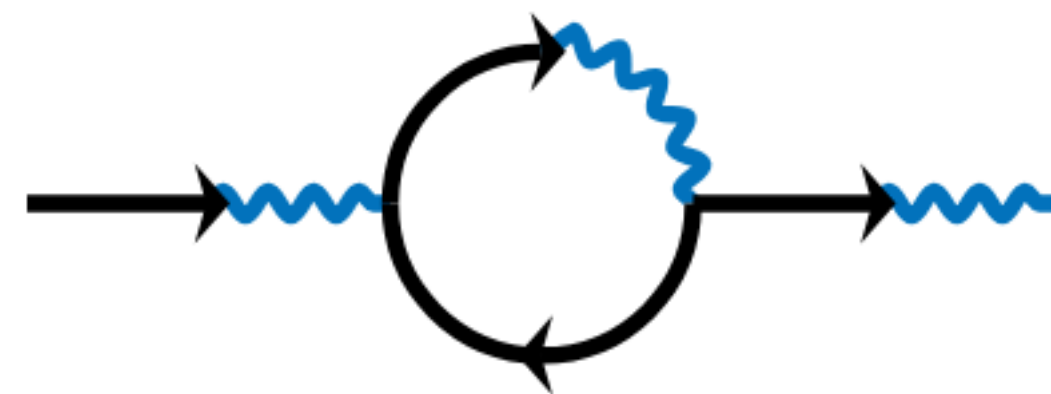
- ▶ Tree-level retarded correlator

$$\longrightarrow \text{wavy line} = \frac{\sigma_0 \mathbf{k}^2}{D \mathbf{k}^2 - i\omega / \mathcal{M}(\omega)}$$

$$\mathcal{M}(\omega) \equiv \frac{2}{\hbar\beta\omega} \tanh\left(\frac{\hbar\beta\omega}{2}\right)$$

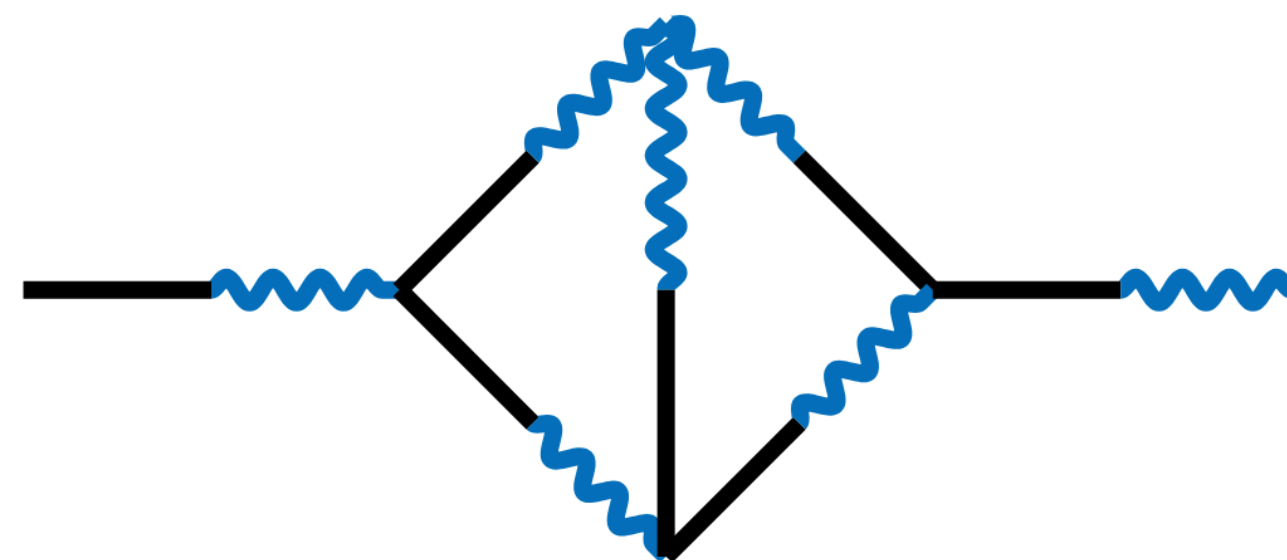
- ▶ One-loop corrections [Chen-Lin, Delacretaz, Hartnoll 2018]

$$D = \sigma_0 \frac{\partial\mu}{\partial n}$$



includes $\hbar^2$ interactions

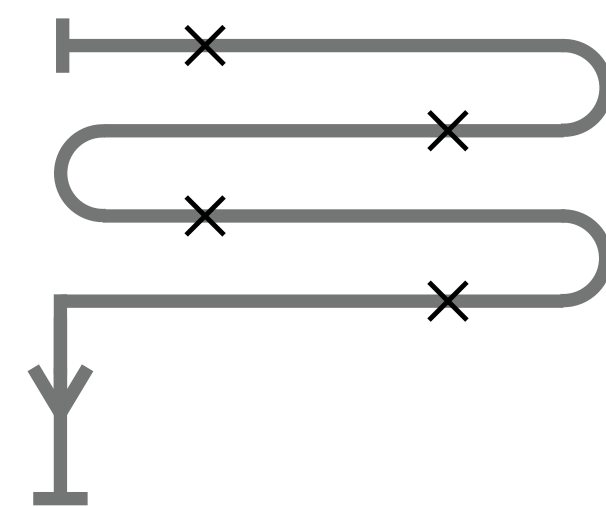
- ▶ Two-loop diagram with “ aaa ” vertex



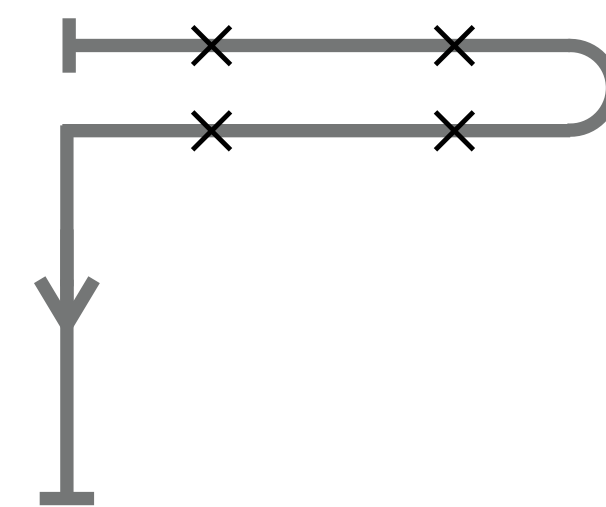
OTOCs and Quantum Chaos

- SK-EFTs for OTOCs require 4-contours.

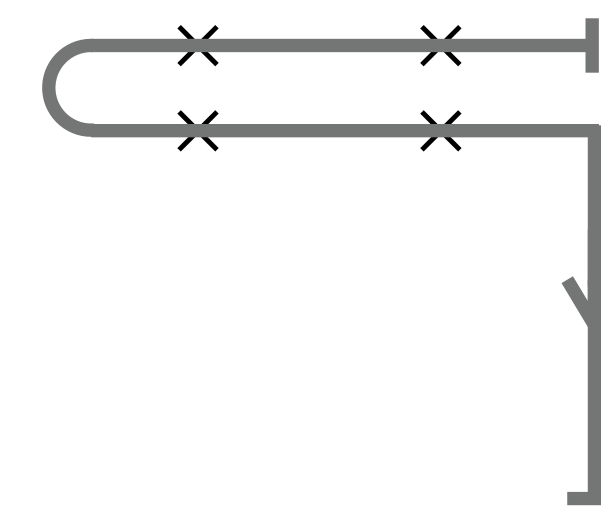
$$\begin{aligned} \langle [W(t), V(0)]^2 \rangle &= \langle W(t) V(0) W(t) V(0) \rangle - \langle V(0) W(t) W(t) V(0) \rangle \\ &\quad + \langle V(0) W(t) V(0) W(t) \rangle - \langle W(t) V(0) V(0) W(t) \rangle \end{aligned}$$



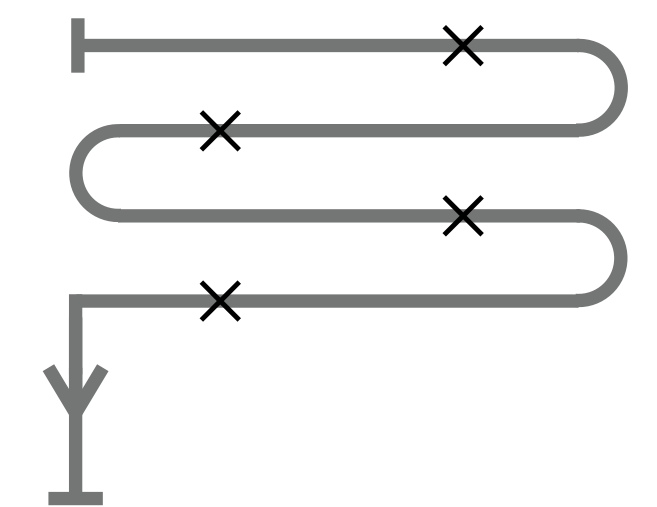
WWVW



-VWWV



-WV VW



VWVW

- We need to find appropriate unitarity conditions and KMS symmetry.

Schwinger-Keldysh Action on 4-Contour

- 4-Contour SK generating functional

$$\mathcal{Z}[A_{1,2,3,4}] = \int \mathcal{D}\phi_{1,2,3,4} \exp\left(i S[\phi_{1,2,3,4}; A_{1,2,3,4}]\right)$$

- Unitarity constraints

$$S_{\text{kkij}} = S_{\text{ikkj}} = S_{\text{ijkk}} = S_{\text{ij}}$$

$$S_{\text{ijkl}}^* = -S_{\text{lkji}}$$

$$\text{Im } S_{\text{ijkl}} \stackrel{?}{\geq} 0$$

$$F_i \equiv F[f_i]$$

$$F_{ij} \equiv F[f_i, f_j]$$

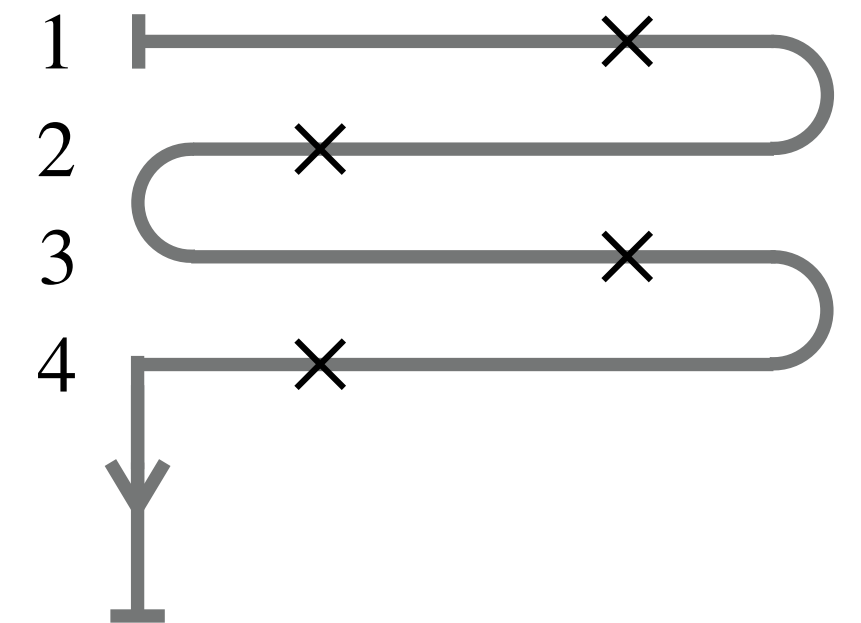
$$F_{ijkl} \equiv F[f_i, f_j, f_k, f_l]$$

$$S_{1234} = \underbrace{S_{12} + S_{23} + S_{34} + S_{14} - S_{13} - S_{24}}_{\text{Hydrostatic + Dissipative (Hydrodynamic)}} + \underbrace{\mathcal{A}_{1234}}_{\text{Chaotic}}$$

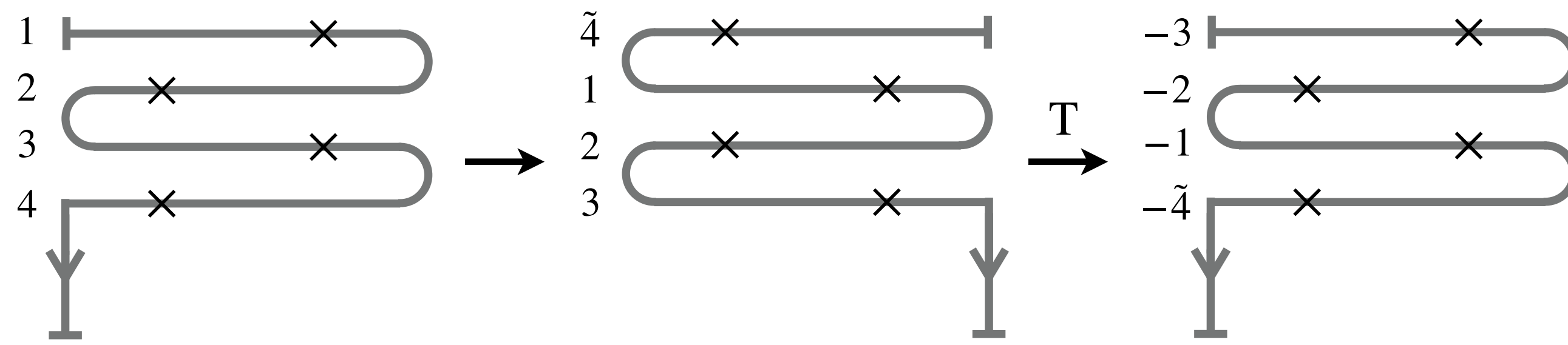
“4-Influence Functional”

Incompatible with KMS

$$\text{e.g. } \sim \cancel{(B_{2t} \ B_{3t})} \cancel{(\mathbf{B}_1 \ \mathbf{B}_2)} \cancel{(\mathbf{B}_3 \ \mathbf{B}_4)}$$



KMS Symmetry on a 4-Contour

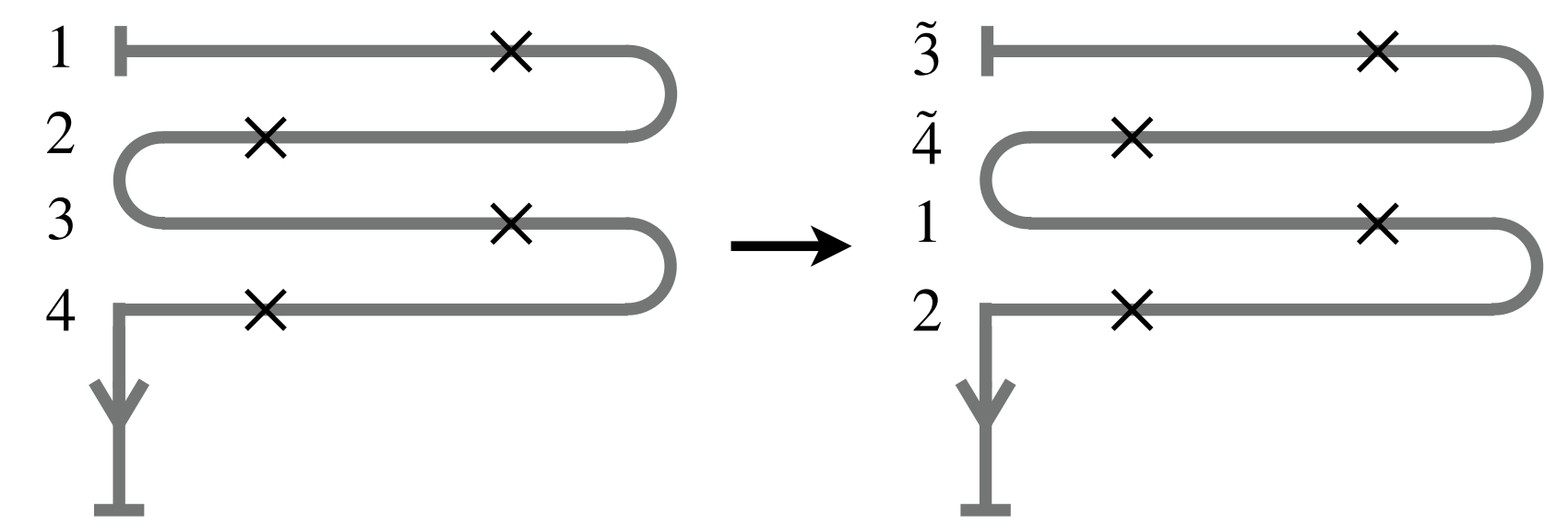


$$Kf_1(t) = \pm f_3(-t)$$

$$Kf_2(t) = \pm f_2(-t)$$

$$Kf_3(t) = \pm f_1(-t)$$

$$Kf_4(t) = \pm f_4(-t - i\hbar\beta)$$



$$Cf_1(t) = f_3(t - i\hbar\beta/2)$$

$$Cf_2(t) = f_4(t - i\hbar\beta/2)$$

$$Cf_3(t) = f_1(t + i\hbar\beta/2)$$

$$Cf_4(t) = f_2(t + i\hbar\beta/2)$$

$$K^2 = 1 \quad C^2 = 1 \quad KCK = C$$

$\longrightarrow \text{Dih}_2$

Implies constraints on $\mathcal{A}_{1234}$

► Generalises to Dih_n for SK-EFTs on $2n$ -contour.

Questions

- Can we construct $\mathcal{A}_{1234}$ compatible with Dih_2 KMS symmetry?
Is there a general prescription for higher-contours?
- Are there extra conditions for maximal chaos? $\text{SL}(2, \mathbb{R})$?
What about near-maximal chaos?
[Blake, Lee, Liu 2018]
[Haehl, Rozali 2018]
- Is there a consistent $\hbar \rightarrow 0$ truncation of SK-EFTs on 4-contour?
If so, what is the stochastic interpretation of “3 noise fields”?
- Does $\mathcal{A}_{1234}$ backreact on retarded correlators? [Jain, Kovtun 2020]
Does it lead to any interesting analytic properties?





THANK YOU

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