

Quantum Fluctuations in Schwinger-Keldysh Hydrodynamics



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[2601.22140] AJ

[2604.xxxxx] Bucca, AJ, Mezei, Milekhin

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EFT and Hydrodynamics

- **Effective field theory (EFT)** is a powerful tool to construct effective macroscopic models for complex many-body systems, largely agnostic of microscopic details.
- Conventional EFTs are only suited to systems at zero temperature or in **thermal equilibrium** (statistical field theory).
- **Hydrodynamics** describes collective macroscopic dynamics of many-body systems operating *close to thermal equilibrium*.
- Hydrodynamics is “*like*” an EFT. It has *effective fields* varying over space and time (e.g. fluid velocity, temperature, density), whose dynamics is dictated by universal symmetry principles. However, it is *not* based on an **effective action principle**.

This has changed recently!

Fluctuations

- Classical hydrodynamics is deterministic and does not account for fluctuations, thermal or quantum.

Fluctuation-Dissipation Theorem (FDT)

$$G^S(\omega, \mathbf{k}) = \hbar \coth\left(\frac{\omega}{2}\hbar\beta\right) \text{Im } G^R(\omega, \mathbf{k})$$

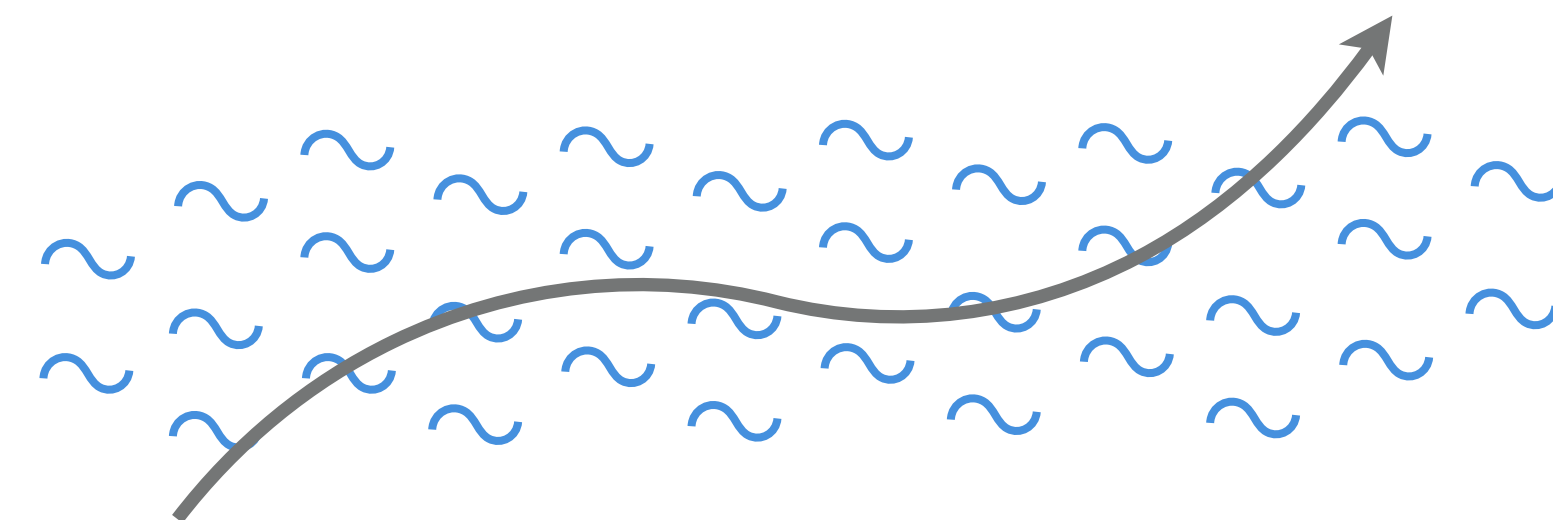
$$\frac{1}{2} \left\langle \{ \mathcal{O}(t_1), \mathcal{O}'(t_2) \} \right\rangle$$

Fluctuation

$$\frac{i}{\hbar} \Theta(t_1 - t_2) \left\langle [\mathcal{O}(t_1), \mathcal{O}'(t_2)] \right\rangle$$

Dissipation

- Fluctuations back-react on hydrodynamics via interactions.



Quantum and Stochastic Fluctuations

$$G^S(\omega, \mathbf{k}) = \hbar \coth\left(\frac{\omega}{2}\hbar\beta\right) \text{Im } G^R(\omega, \mathbf{k})$$

$$\frac{2}{\beta\omega} \left(1 + \frac{1}{12}(\hbar\beta\omega)^2 + \frac{1}{720}(\hbar\beta\omega)^4 + \dots \right)$$

- **Fluctuations** survive in the classical limit $\omega \ll 1/t_{\text{micro}} \ll 1/(\hbar\beta)$.
These are accounted for by **stochastic hydrodynamics**.

$$G_{nn}^R(\omega, \mathbf{k}) = \frac{\mathbf{k}^2\sigma}{D\mathbf{k}^2 - i\omega} - \frac{i\omega\mathbf{k}^4\#}{(D\mathbf{k}^2 - i\omega)^2} \left(\mathbf{k}^2 - \frac{2i\omega}{D} \right)^{\frac{d}{2}-1} + \dots \quad \text{“Long-time tails”}$$

$$\left. \frac{n(t, \mathbf{k})}{n(0, \mathbf{k})} \right|_{t \gg 1/(D\mathbf{k}^2)} = e^{-D\mathbf{k}^2 t} + \frac{\#}{t^{d/2}} e^{-\frac{1}{2}D\mathbf{k}^2 t} + \dots$$

[De Schepper, Van Beyeren, Ernst 1974]

[Forster, Nelson, Stephen 1977]

[Arnold, Yaffe 1997]

[Kovtun, Yaffe 2003]

[Chen-Lin, Delacretaz, Hartnoll 2018]

- Quantum fluctuations become important when $t_{\text{micro}} \gtrsim \hbar\beta$.

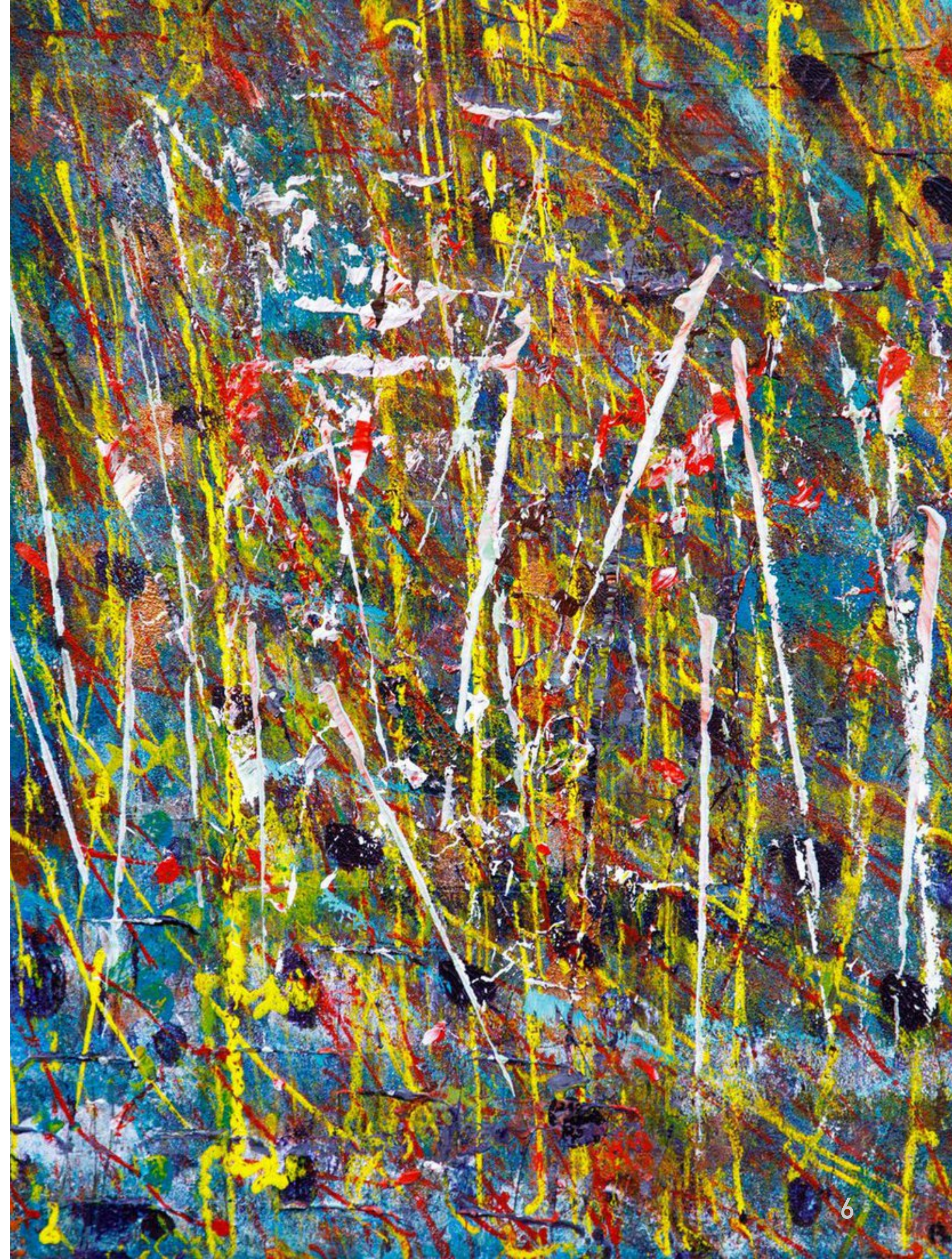


Schwinger-Keldysh Hydrodynamics

- **Schwinger-Keldysh (SK) hydrodynamics** is an EFT completion of classical hydrodynamics, which systematically incorporates fluctuations, both stochastic and quantum. [Kovtun, Moore, Romatschke 2014]
[Harder, Kovtun, Ritz 2015]
[Crossley, Glorioso, Liu 2015]
[Haehl, Loganayagam, Rangamani 2015]
[Jensen, Pinzani-Fokeeva, Yarom 2017]
- It contains new **stochastic transport coefficients**, invisible in classical or stochastic hydrodynamics. [AJ, Kovtun 2020]
- It can be used beyond thermal equilibrium. [Jain, Kovtun, Ritz, Shukla (2020)]
[Landry (2023)]
[Lin, Bu, Lie (2023)]
[Armas, AJ, Lier (2024)]
- **Goal of today:** Inspect the effects of **quantum fluctuations** arising from SK hydrodynamics in the simplest calculable example of an Abelian diffusive conserved charge.

Dissipation and Fluctuations

Lightening review of hydrodynamics



- Hydrodynamics of a conserved scalar n , with conservation equation

$$\partial_t n + \nabla \cdot \mathbf{J} = 0$$

Given $\mathbf{J} = \mathbf{J}(n)$, this equation governs the classical evolution of n .

- The **constitutive relations** are given as

$$\Rightarrow \quad \mathbf{J} = -\sigma(n) \left(\nabla \mu(n) - \mathbf{E} \right) + O(\nabla^2)$$

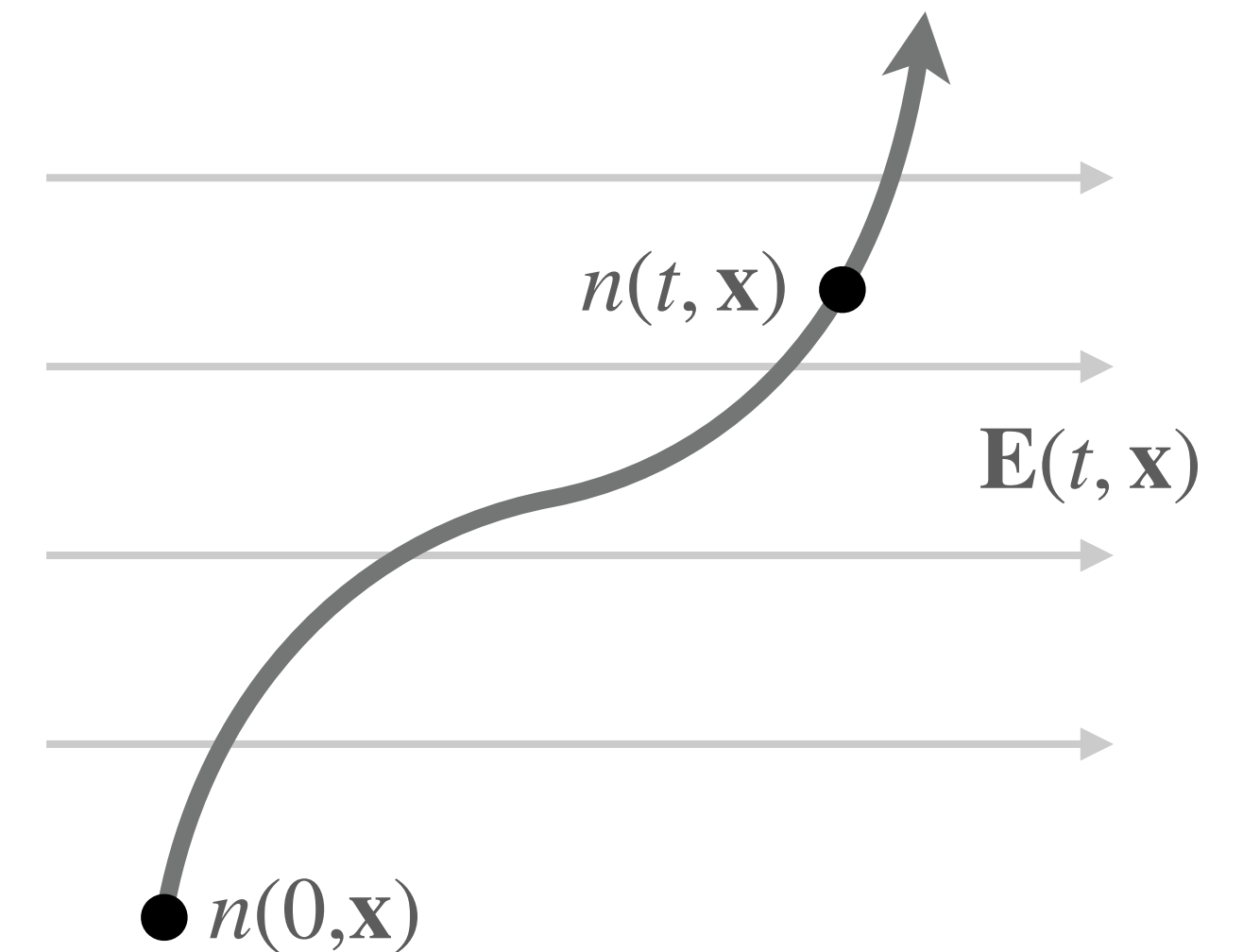
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conductivity
chemical potential

- Hydrodynamic prescription for **retarded correlators** [Kadanoff, Martin 1963]

$$G^{\text{R}}(t_1 - t_2) = \frac{\delta}{\delta A(t_2)} J_{\text{onshell}}(t_1)$$

No fluctuations: **incompatible with FDT**



$\mathbf{E} = \nabla A_t - \partial_t \mathbf{A}$ is applied electric field

$$J^\mu = (n, \mathbf{J})$$

$$\mathbf{A}_\mu = (\mathbf{A}_t, \mathbf{A})$$

Fluctuations

- Hydrodynamic observables have variance due to thermal noise

$$\langle JJ \dots \rangle \neq \langle J \rangle \langle J \rangle \dots$$

- We introduce **random thermal noise** [Martin, Siggia, Rose (1973)]

$$\mathbf{J} = -\sigma (\nabla \mu - \mathbf{E}) + \boldsymbol{\theta}$$

$$\exp W[A, A_a] = \int \mathcal{D}\boldsymbol{\theta} \exp \left(i \int_x J[A, \boldsymbol{\theta}] A_a \right) \mathcal{P}[\boldsymbol{\theta}]$$

drawn from some **probability distribution** $\mathcal{P}[\boldsymbol{\theta}]$.

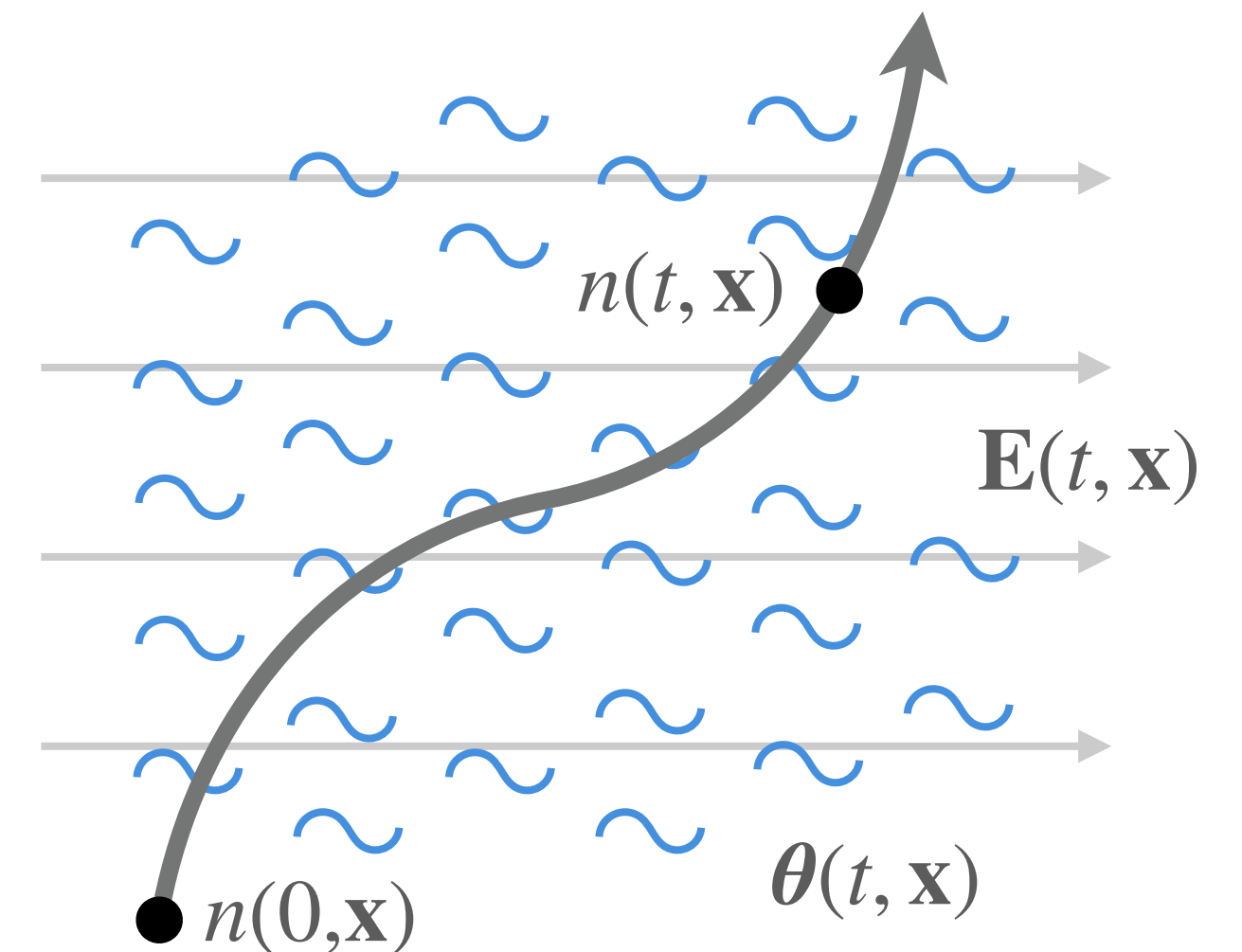
$$G^R = \frac{\delta}{\delta A} \frac{-i\delta}{\delta A_a} W[A, A_a] \Big|_{\delta A, A_a=0}$$

$$G^S = \frac{-i\delta}{\delta A_a} \frac{-i\delta}{\delta A_a} W[A, A_a] \Big|_{\delta A, A_a=0}$$

engineered to satisfy **FDT**

$$\ln \mathcal{P}[\boldsymbol{\theta}] = - \int_x \frac{1}{4T\sigma(n)} \boldsymbol{\theta}^2 + \dots$$

non-universal



Effective Action for Fluctuations

- Derive an effective action for fluctuations

$$\begin{aligned}\exp W[A, A_a] &= \int \mathcal{D}\theta \exp \left(i \int_x J[A, \theta] A_a \right) \mathcal{P}[\theta] \\ &= \int \mathcal{D}\theta \mathcal{D}\phi_a \mathcal{D}n \exp \left(i \int_x -\phi_a \partial_\mu J^\mu + J^\mu A_{a\mu} \right) \mathcal{P}[\theta] \\ &= \int \mathcal{D}\phi_a \mathcal{D}n \exp \left(iS[n, \phi_a; A, A_a] \right)\end{aligned}$$

- MSR effective action [Martin, Siggia, Rose (1973)]

$$S = \int_x n B_{at} - \sigma(n) \left(\nabla \mu(n) - \mathbf{E} \right) \cdot \mathbf{B}_a + iT\sigma(n) \mathbf{B}_a \cdot \mathbf{B}_a + \dots$$

$$B_a = \partial \phi_a + A_a$$

$$\frac{\delta S}{\delta n} = 0 \implies \varphi_a = 0$$

$$\frac{\delta S}{\delta \phi_a} = 0 \implies \partial_\mu J^\mu = 0$$

(for $A_a = 0$)

Unitarity Constraints and Classical KMS Symmetry

- SK language: Dynamical fields: ϕ_1, ϕ_2 . Background fields $A_{1\mu}, A_{2\mu}$.

Classical field: $\mu(n) \equiv B_{rt} + \mu_0$ Noise field: ϕ_a

Classical source: $A_\mu \equiv A_{r\mu}$ Noise source: $A_{a\mu}$

$$\mathcal{L} = n(\mu) B_{at} - \sigma(\mu) \partial_t \mathbf{B}_r \cdot \mathbf{B}_a + iT\sigma(\mu) \mathbf{B}_a \cdot \mathbf{B}_a + \dots$$

$$f_r = \frac{f_1 + f_2}{2}$$

$$f_a = \frac{f_1 - f_2}{\hbar}$$

$$B = \partial\phi + A$$

- **Unitarity constraints:**

$$S[f_1 = f, f_2 = f] = 0$$

$$S[f_2, f_1] = -S^*[f_1, f_2]$$

$$\text{Im } S[f_1, f_2] \geq 0.$$

- **Classical Kubo-Martin-Schwinger (KMS) symmetry:**

$$f_r(t) \leftrightarrow \eta_f f_r(-t)$$

$$f_a(t) \leftrightarrow \eta_f \left(f_a(-t) + i\beta \partial_t f_r(-t) \right)$$

time-reversal*
eigenvalue

$$G^S(\omega, \mathbf{k}) = \frac{2}{\beta\omega} \text{Im } G^R(\omega, \mathbf{k})$$

- Global symmetries: $\delta\phi_{1,2} = -\Lambda_{1,2}$, $\delta A_{1,2} = d\Lambda_{1,2}$. Reparametrisations: $\delta\phi_{1,2} = \lambda(\mathbf{x})$

*This assumes that the microscopic system realises T symmetry. Analogous versions exist for PT, CT, or CPT.

Schwinger-Keldysh Effective Action

- Schwinger-Keldysh generating functional on a **closed-time contour**

$$\exp W[A_{1,2}] = \int \mathcal{D}\Psi_{1,2} \exp \left(i S_{\text{micro}}[\Psi_1; A_1] - i S_{\text{micro}}[\Psi_2; A_2] \right)$$

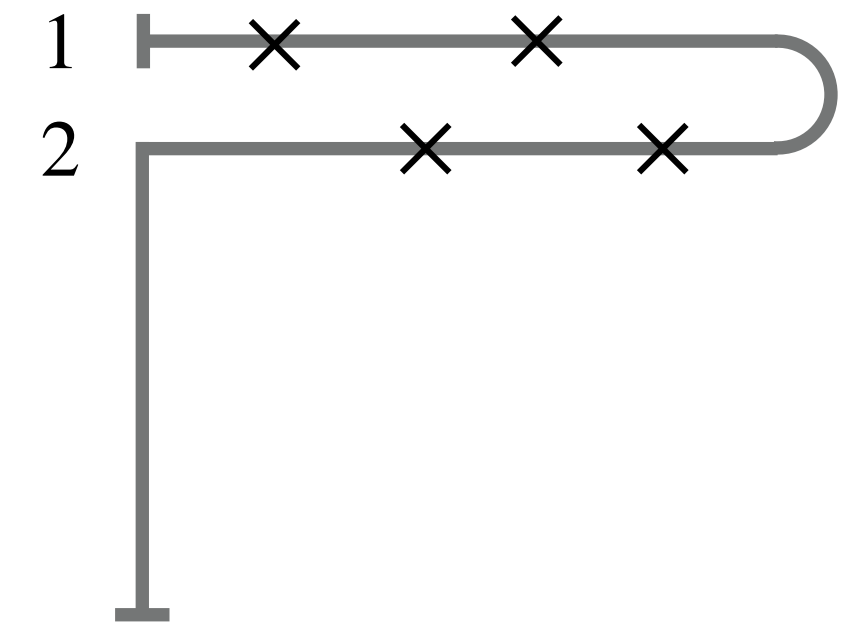
↓ RG

$$\exp W[A_{1,2}] = \int \mathcal{D}\phi_{1,2} \exp \left(i S[\phi_{1,2}; A_{1,2}] \right)$$

- RG generates non-trivial coupling terms in the action, which may be imaginary

$$S_{12} = \underbrace{S_1 - S_2}_{\text{thermodynamic}} + \underbrace{\mathcal{A}_{12}}_{\text{dissipative} + i \text{ fluctuations}}$$

“Influence Functional”



$$F_i \equiv F[f_i]$$

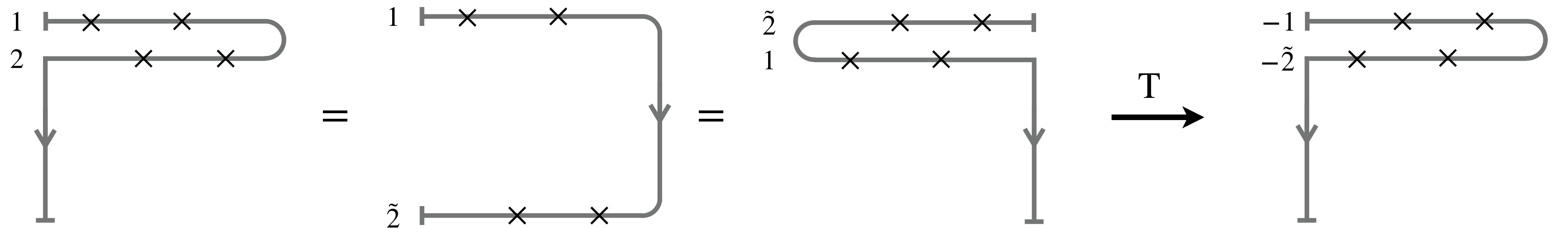
$$F_{ij} \equiv F[f_i, f_j]$$

- Unitarity constraints follow from general properties of the closed-time contour.

$$S_{ii} = 0 \qquad S_{ij}^* = -S_{ji} \qquad \text{Im } S_{ij} \geq 0$$

KMS Symmetry

- FDT is non-linearly realised by the **KMS symmetry**



$$\tilde{\mathbf{i}} : f_{\mathbf{i}}(t - i\hbar\beta)$$

$$-\mathbf{i} : \eta_f f_{\mathbf{i}}(-t)$$

$$\mathbf{K} f_1(t) = \eta_f f_1(-t)$$

$$\mathbf{K} f_r(t) = \eta_f f_r(-t) + O(\hbar)$$

$$\begin{aligned} \mathbf{K} f_2(t) &= \eta_f f_2(-t - i\hbar\beta) \\ &= \eta_f e^{-i\hbar\beta \partial_t} f_2(-t) \end{aligned}$$

$\implies$

$$\mathbf{K} f_a(t) = \eta_f \left(f_a(-t) + i\beta \partial_t f_r(-t) \right) + O(\hbar)$$

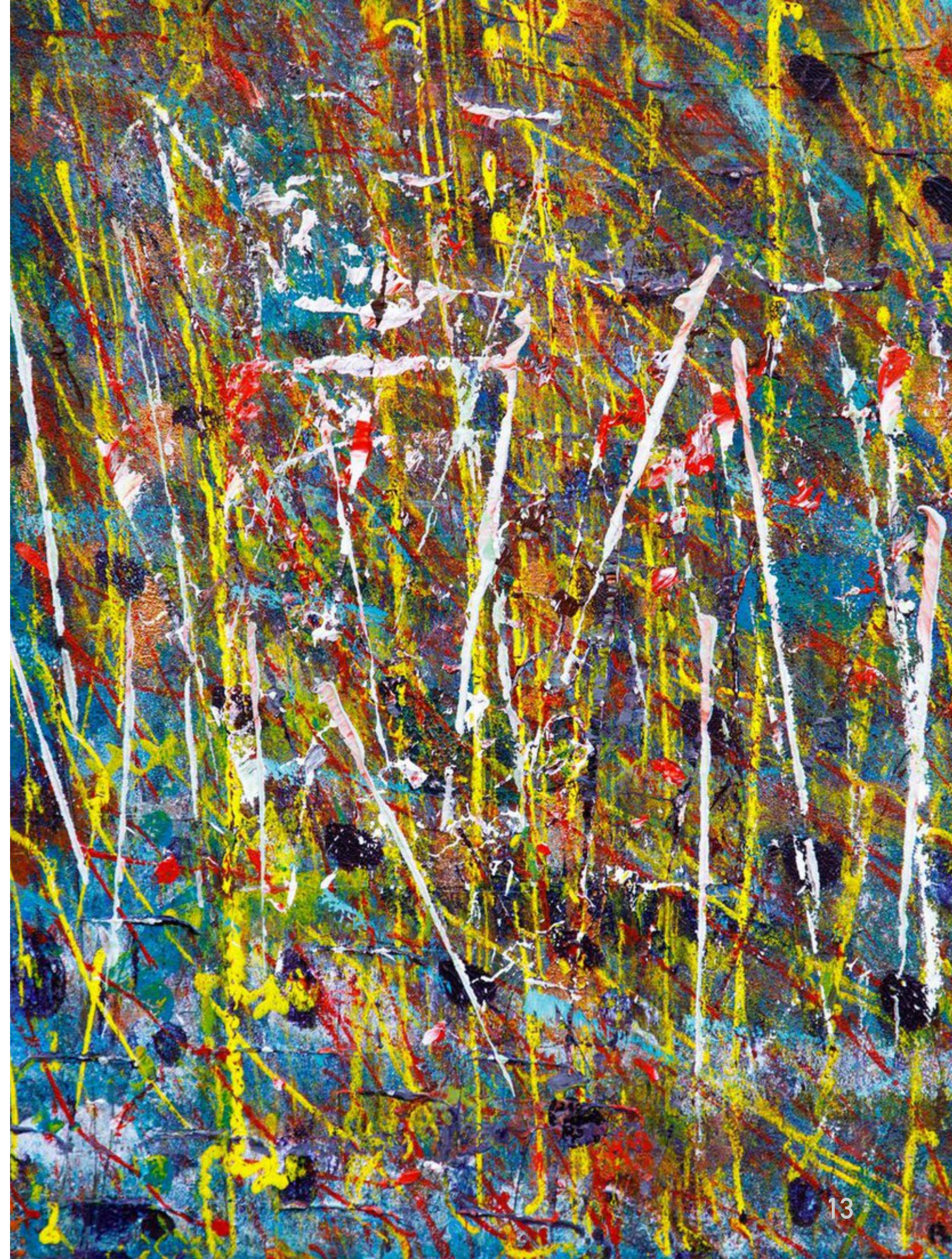
$$f_r = \frac{f_1 + f_2}{2}$$

$$f_a = \frac{f_1 - f_2}{\hbar}$$

$$\mathbf{K}^2 = 1$$

Quantum Corrections

in SK hydrodynamics



Quantum-Complete SK Effective Action – Free Theory

- The **free** SK effective action

$$\mathcal{L}_{\text{free}} = n_0 B_{at} + \chi_0 B_{rt} B_{at} - \sigma_0 \partial_t \mathbf{B}_r \cdot \mathbf{B}_a + iT\sigma_0 \mathbf{B}_a \cdot \mathbf{B}_a + \dots$$

$$n(\mu) = n_0 + \chi_0 (\mu - \mu_0) + \dots$$

$$\sigma(\mu) = \sigma_0 + \dots$$

- This can be made **quantum complete** by simply upgrading the noise piece

$$\mathcal{L}_{\text{free}} = n_0 B_{at} + \chi_0 B_{rt} B_{at} - \sigma_0 \partial_t \mathbf{B}_r \cdot \mathbf{B}_a + \boxed{iT\sigma_0 \mathbf{B}_a \cdot \mathfrak{D} \mathbf{B}_a} + \dots$$

$$\begin{aligned} \mathfrak{D} &= \frac{\hbar\beta\partial_t}{2} \cot\left(\frac{\hbar\beta\partial_t}{2}\right) \\ &= 1 - \frac{\hbar^2\beta^2}{12} \partial_t^2 - \frac{\hbar^4\beta^4}{720} \partial_t^4 + \dots \end{aligned}$$

$$G^S(\omega, \mathbf{k}) = \frac{2}{\beta} \frac{\mathbf{k}^2 \sigma}{|D\mathbf{k}^2 - i\omega|^2} \longrightarrow \hbar\omega \coth\left(\frac{\hbar\beta\omega}{2}\right) \frac{\mathbf{k}^2 \sigma}{|D\mathbf{k}^2 - i\omega|^2}$$

- The effective action is local perturbatively in derivatives.
- It satisfies the **fully-quantum KMS symmetry**.

$$G^S(\omega, \mathbf{k}) = \hbar \coth\left(\frac{\omega}{2} \hbar\beta\right) \text{Im } G^R(\omega, \mathbf{k})$$

Interactions – Thermodynamic

- Promoting the non-linear thermodynamic part to quantum level is quite simple.
- The **free thermodynamic part** can be cast as

$$n_0 B_{at} + \chi_0 B_{rt} B_{at} = \frac{1}{\hbar} \left(p_0 + n_0 B_{1t} + \frac{\chi_0}{2} B_{1t}^2 \right) - \frac{1}{\hbar} \left(p_0 + n_0 B_{2t} + \frac{\chi_0}{2} B_{2t}^2 \right) \quad S_{12} \sim \boxed{S_1 - S_2}$$

- Correspondingly, non-linearly we can write

$$n(\mu) B_{at} \longrightarrow \frac{p(\mu_1) - p(\mu_2)}{\hbar} \quad p'(\mu) = n(\mu) \quad \mu_{1,2} \equiv B_{1,2t} + \mu_0$$

It is trivially invariant under independent KMS transformations in “1” and “2” sectors.

- At cubic orders in fields, this gives

$$\mathcal{L} \sim \frac{\chi_1}{2} B_{rt}^2 B_{at} + \frac{\hbar^2 \chi_1}{24} B_{at}^3 + \dots \quad n(\mu) = n_0 + \chi_0 (\mu - \mu_0) + \frac{\chi_1}{2} (\mu - \mu_0)^2 + \dots$$

Interactions – Dissipative and Noise

- The non-linear dissipative and noise part is more involved.
- Sparing the derivation, we find

$$\begin{aligned} \mathcal{L} \sim & -\sigma(\mu)_r \partial_t \mathbf{B}_r \cdot \mathbf{B}_a + iT\sigma(\mu)_r \mathbf{B}_a \cdot \mathfrak{D}\mathbf{B}_a \\ & + iT\mathbf{B}_r \cdot \left(\sigma(\mu)_a \mathfrak{D}\mathbf{B}_a - \mathfrak{D}\sigma(\mu)_a \mathbf{B}_a \right) - \frac{\hbar^2}{8} \partial_t \sigma(\mu)_a \mathbf{B}_a^2 \end{aligned}$$

- This respects all SK requirements, including quantum KMS symmetry.
- At cubic orders in fields, this gives

$$\begin{aligned} \mathcal{L} \sim & -\sigma_1 B_{rt} \mathbf{B}_a \cdot \partial_t \mathbf{B}_r - \frac{\hbar^2 \sigma_1}{8} \mathbf{B}_a^2 \partial_t B_{at} \\ & + iT\sigma_1 \left(B_{rt} \mathbf{B}_a \cdot \mathfrak{D}\mathbf{B}_a + B_{at} \mathbf{B}_r \cdot \mathfrak{D}\mathbf{B}_a - \mathbf{B}_r \cdot \mathbf{B}_a \mathfrak{D}B_{at} \right) + \dots \end{aligned}$$

$$S_{12} \sim \boxed{\mathcal{A}_{12}}$$

$$\begin{aligned} F(f)_r &\equiv \frac{F(f_1) + F(f_2)}{2} \\ &= F(f_r) + \frac{\hbar^2}{8} f_a^2 F''(f_r) + \dots \end{aligned}$$

$$\begin{aligned} F(f)_r &\equiv \frac{F(f_1) - F(f_2)}{\hbar} \\ &= f_a F'(f_r) + \frac{\hbar^2}{24} f_a^3 F'''(f_r) + \dots \end{aligned}$$

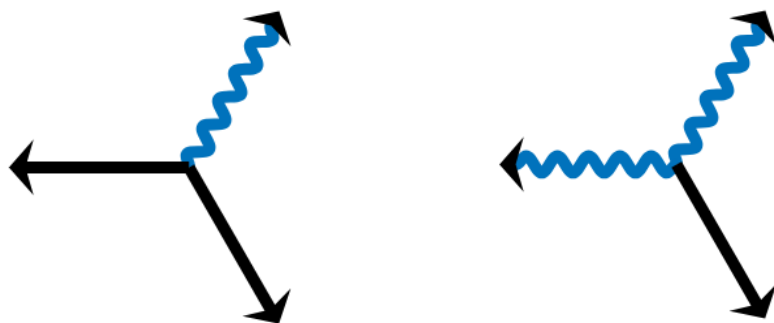
$$\sigma(\mu) = \sigma_0 + \sigma_1 (\mu - \mu_0) + \dots$$

Summary

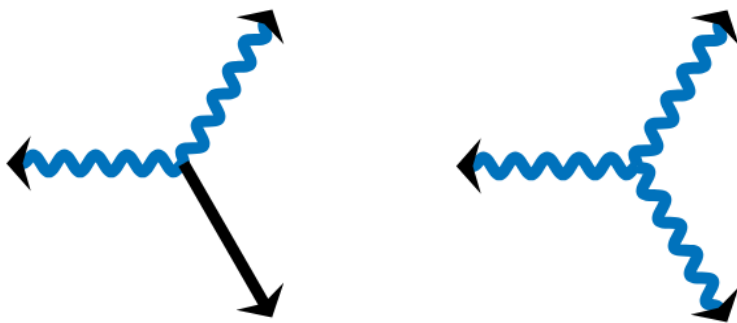
S_{12}	$S_1 - S_2$	$\mathcal{A}_{12}$ (Dissipative)	$\mathcal{A}_{12}$ (i Fluctuations)
Free Action	$\chi_0 B_{at} B_{rt}$	$-\sigma_0 \partial_t \mathbf{B}_r \cdot \mathbf{B}_a$	$iT\sigma_0 \mathbf{B}_a \cdot \mathfrak{D}\mathbf{B}_a$
3-point Interactions	$\frac{\chi_1}{2} B_{at} B_{rt}^2 + \frac{\hbar^2 \chi_1}{24} B_{at}^3$	$-\sigma_1 B_{rt} \mathbf{B}_a \cdot \partial_t \mathbf{B}_r - \frac{\hbar^2 \sigma_1}{8} \mathbf{B}_a^2 \partial_t B_{at}$	$iT\sigma_1 B_{rt} \mathbf{B}_a \cdot \mathfrak{D}\mathbf{B}_a + iT\sigma_1 B_{at} \mathbf{B}_r \cdot \mathfrak{D}\mathbf{B}_a$ $-iT\sigma_1 \mathbf{B}_r \cdot \mathbf{B}_a \mathfrak{D}B_{at}$



Free Propagators



Stochastic Interactions

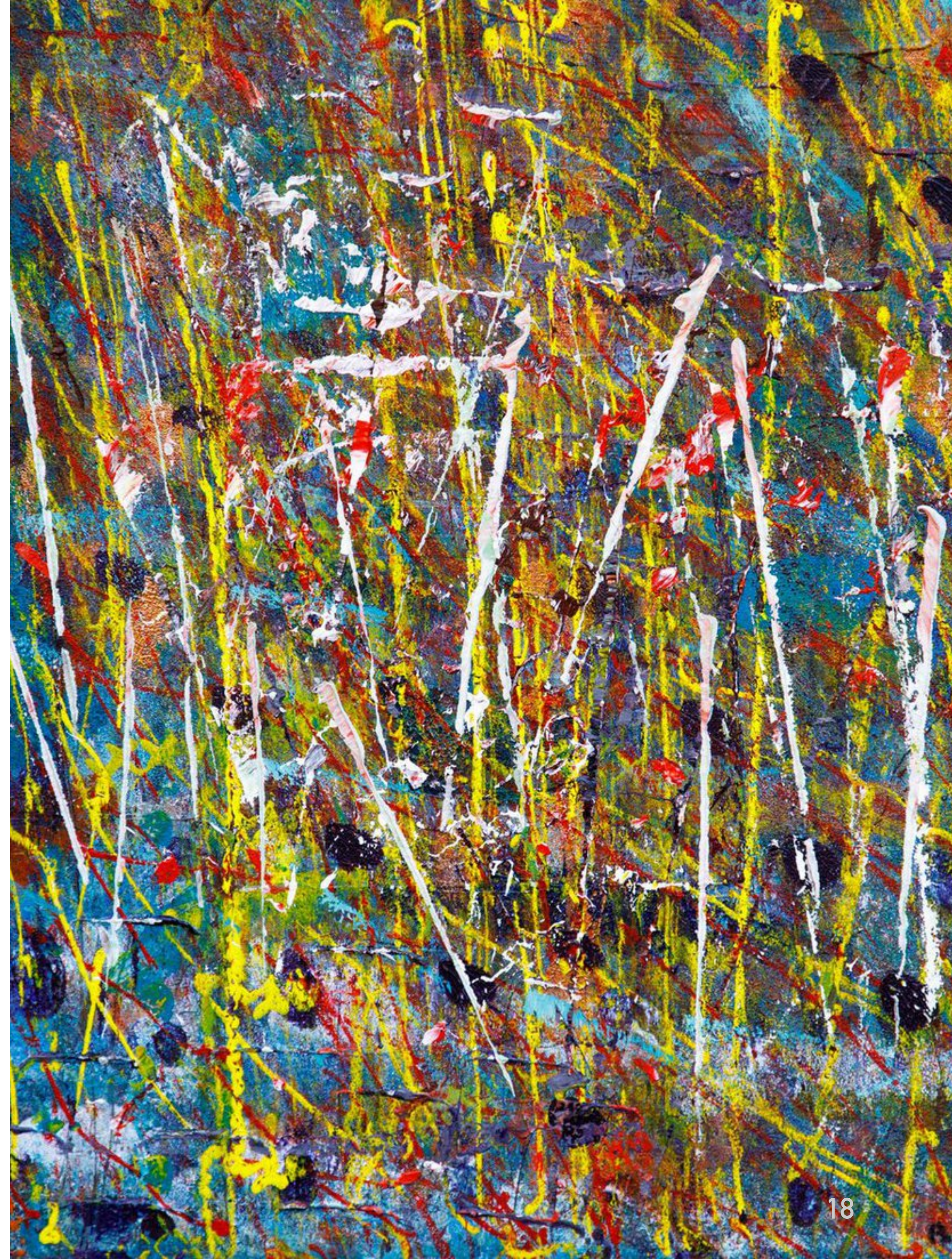


Quantum Interactions



Quantum Fluctuations

in retarded correlation functions



Stochastic Fluctuations in Retarded Correlators

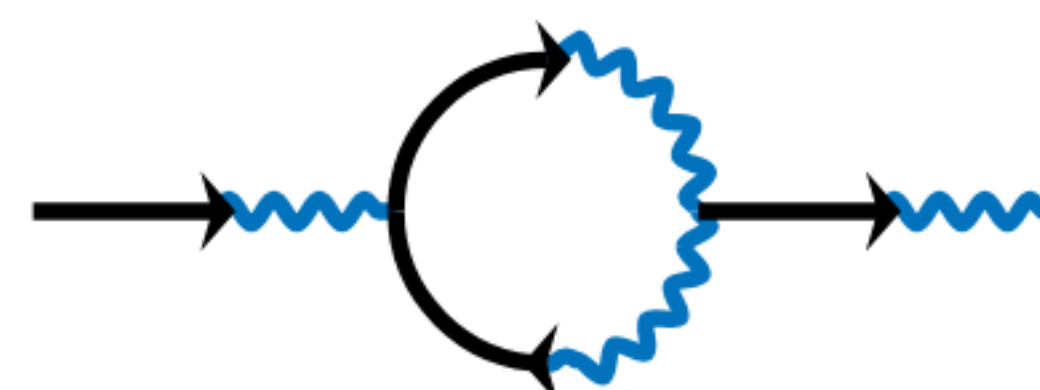
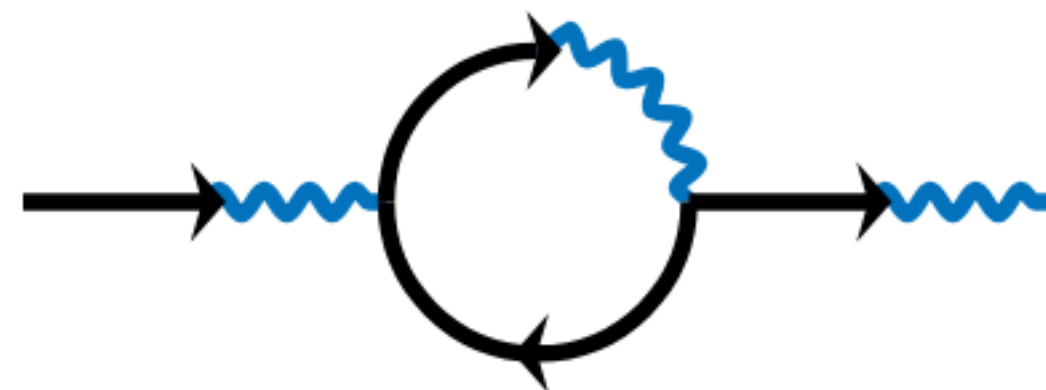
- Tree-level retarded correlator:



$$G_{nn}^R(\omega, \mathbf{k}) = \frac{\mathbf{k}^2 \sigma}{D \mathbf{k}^2 - i\omega} + \dots$$

$$D = \frac{\sigma}{\partial n / \partial \mu}$$

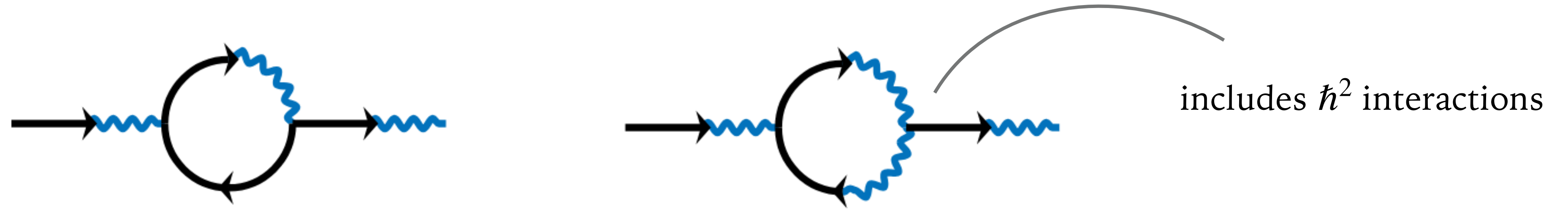
- One-loop stochastic fluctuations [Chen-Lin, Delacretaz, Hartnoll 2018]



$$G_{nn}^R(\omega, \mathbf{k}) = \frac{\mathbf{k}^2 \sigma}{D \mathbf{k}^2 - i\omega} + \frac{i\omega \mathbf{k}^4}{(D \mathbf{k}^2 - i\omega)^2} \frac{(\partial D / \partial \mu)^2}{\beta D} \frac{\Gamma(1 - d/2)}{(16\pi)^{d/2}} \left(\mathbf{k}^2 - \frac{2i\omega}{D} \right)^{\frac{d}{2}-1} + \dots$$

Quantum Fluctuations in Retarded Correlators

► One-loop quantum fluctuations



$$G_{nn}^R(\omega, \mathbf{k}) = \frac{\mathbf{k}^2 \sigma}{D \mathbf{k}^2 - i\omega} + \frac{i\omega \mathbf{k}^4}{(D \mathbf{k}^2 - i\omega)^2} \frac{(\partial D / \partial \mu)^2}{\beta D} \frac{\Gamma(1 - d/2)}{(16\pi)^{d/2}} \left(\mathbf{k}^2 - \frac{2i\omega}{D} \right)^{\frac{d}{2}-1} Z(\omega, \mathbf{k})$$

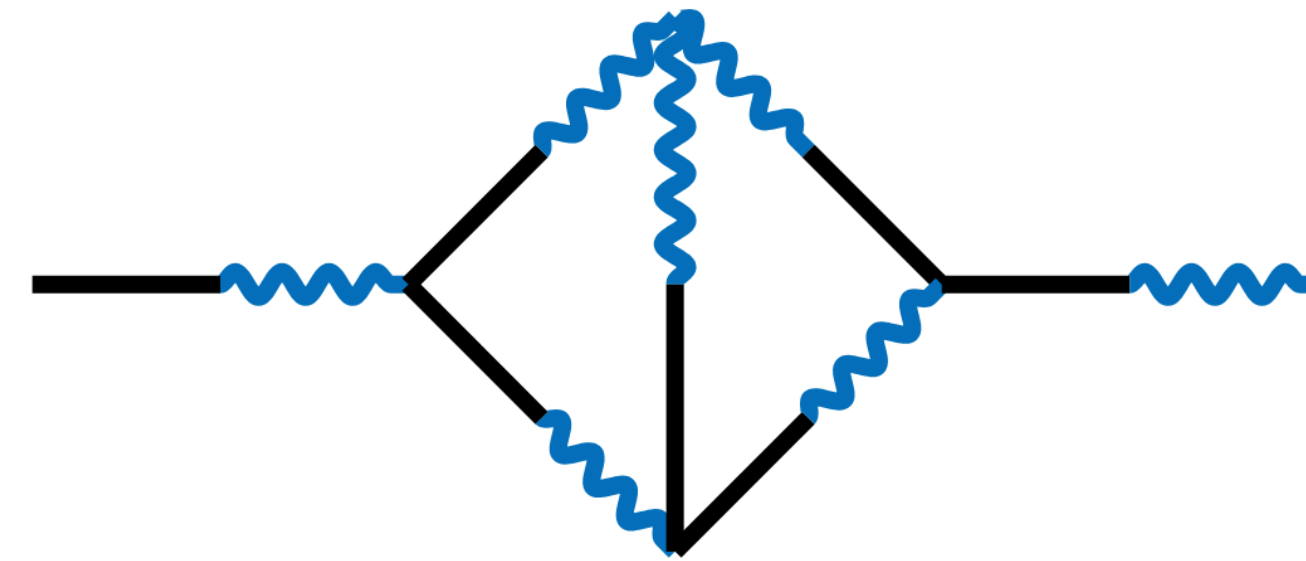
$$Z(\omega, \mathbf{k}) = \left(1 - 2 \sum_{n=1}^{\infty} \zeta(2n) \frac{\hbar \beta^{2n}}{8\pi} (D \mathbf{k}^2 - 2i\omega)^{2n} \mathcal{P}_{d,2n} \left(\frac{\mathbf{k}^2}{\mathbf{k}^2 - 2i\omega/D} \right) \right) = \frac{\hbar \beta \omega}{4} \coth \left(\frac{\hbar \beta \omega}{4} \right) + O(D \mathbf{k}^2 / \omega)$$

$$\mathcal{P}_{d,2n}(z) = -2^{2n-1} (z-1)^{2n} \sum_{m=1}^{2n-1} \binom{2n-1}{m-1} \left(\frac{1}{2^{m-1}} \frac{\mathcal{J}_{d,m}(z)}{(z-1)^m} + \frac{1}{2^m} \frac{\mathcal{J}_{d,m+1}(z)}{(z-1)^{m+1}} \right)$$

$$\mathcal{J}_{d,m}(\mathbf{k}^2) = \frac{(4\pi)^{d/2} \Gamma(d/2)}{\Gamma(1 - d/2 - m) \Gamma(d/2 + m)} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{(\mathbf{q} + \mathbf{k})^{2m}}{\mathbf{q}^2 + 1}$$

Quantum Interactions

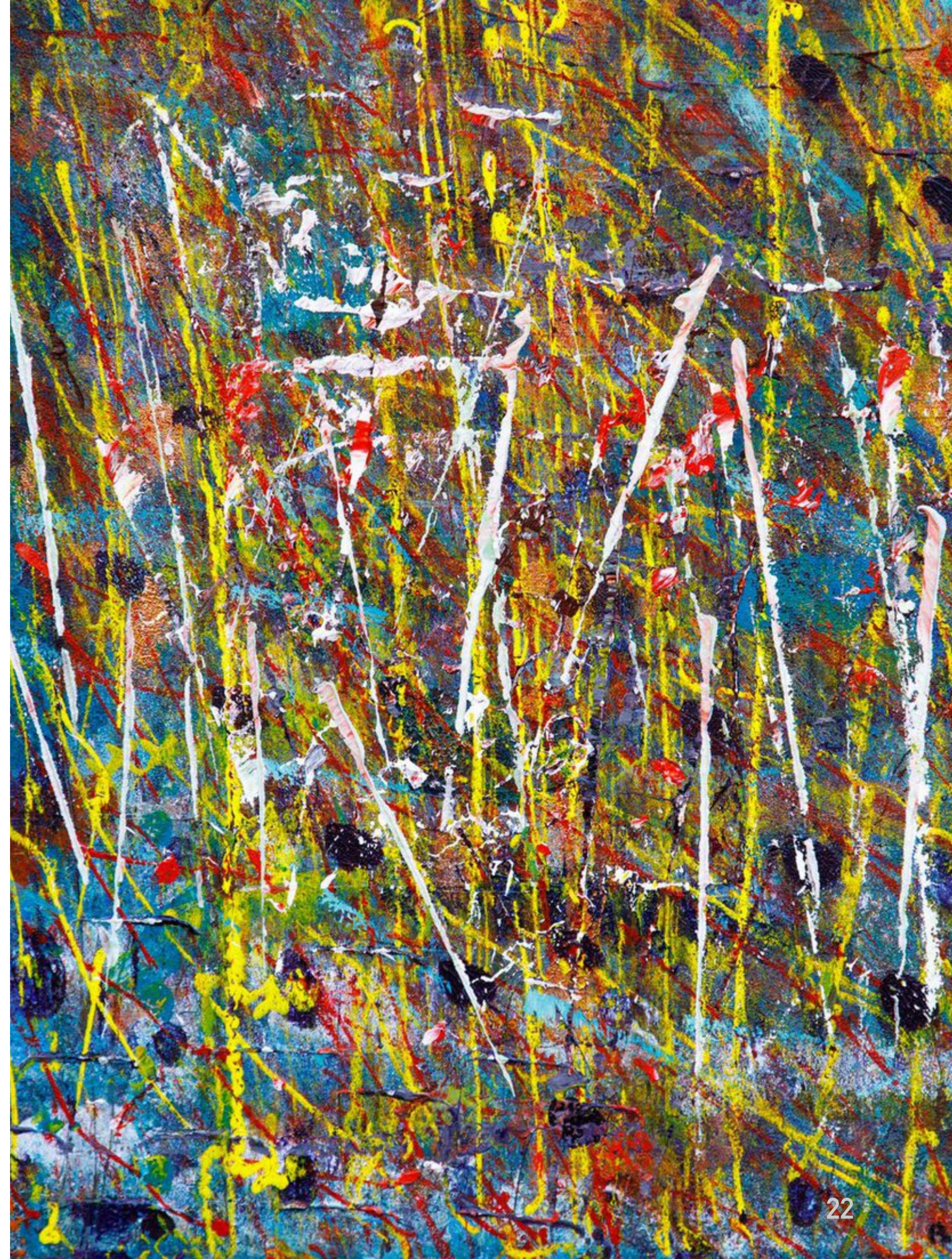
- Starting at two-loops, the retarded correlators become sensitive to quantum interactions



- Could these lead to new analytic structures?

Outlook

and questions



SYK Chain

- Sachdev–Ye–Kitaev (SYK) energy with nearest neighbour interactions [Maldacena, Stanford 2016]

$$H = \sum_x \sum_{i,j,\dots} J_0^{i,j,\dots} \left(\Psi_i(x) \Psi_j(x) \dots \right) + \sum_x \sum_{i,j,\dots;k,l,\dots} J_1^{i,j,\dots;k,l,\dots} \left(\Psi_i(x) \Psi_j(x) \dots \right) \left(\Psi_k(x+a) \Psi_l(x+a) \dots \right)$$

$$\left\langle J_0^{i,j,\dots} \right\rangle = 0 \quad \left\langle \left(J_0^{i,j,\dots} \right)^2 \right\rangle = J_0^2 \quad \left\langle J_1^{i,j,\dots;k,l,\dots} \right\rangle = 0 \quad \left\langle \left(J_1^{i,j,\dots;k,l,\dots} \right)^2 \right\rangle = J_1^2$$

- The low-energy description is given by an **interacting Schwarzian theory** [Maldacena, Qi 2018]

[Altland, Bagrets, Kamenev 2019]

[Almheiri, Milekhin, Swingle 2019]

[Bucca, Mezei 2024]

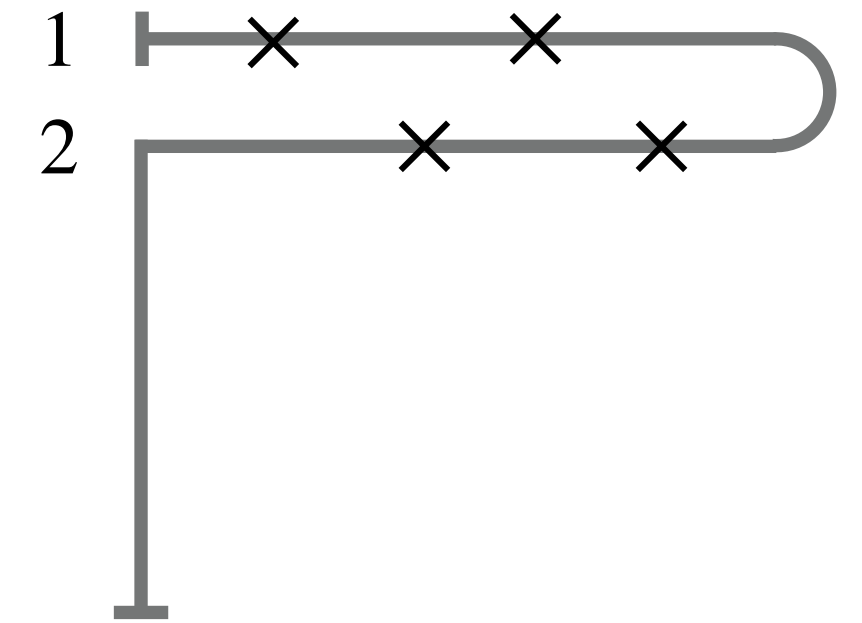
$$S_{\text{SYK-Chain}} = -\mathcal{N} \int dt dx \text{Sch}[\tanh(\pi F(t, x)), t] \\ -i\mathcal{N}' \int dt dt' \left(\frac{\partial_t F(t, x) \partial_{t'} F(t', x)}{\sinh^2(\pi F(t, x) - \pi F(t', x))} \right)^{1/2} \left(\frac{\partial_t F(t, x+a) \partial_{t'} F(t', x+a)}{\sinh^2(\pi F(t, x+a) - \pi F(t', x+a))} \right)^{1/2}$$

SK hydrodynamics for SYK Chain

- SYK chain theory on a **closed-time contour**: [Bucca, AJ, Mezei, Milekhin 2026]

$$S_{\text{SYK-Chain}} = \int dt L_{\text{Sch}}[F(t)] + \int dt dt' L_{\text{int}}[F(t), F(t')]$$

$$S = \int dt \left(L_{\text{Sch}}[F_1(t)] - L_{\text{Sch}}[F_2(t)] \right) + \int dt dt' \left(L_{\text{int}}[F_1(t), F_1(t')] - L_{\text{int}}[F_1(t), F_2(t')] - L_{\text{int}}[F_2(t), F_1(t')] + L_{\text{int}}[F_2(t), F_2(t')] \right)$$



- It is described by the **SK hydrodynamics for energy diffusion**

$$\mathcal{L} = -\epsilon(T) \partial_t \phi_a + \kappa(T) \nabla \phi_a \cdot \nabla T + iT^2 \kappa(T) (\nabla \phi_a)^2 + \dots \quad \partial_t \epsilon(T) + \nabla \cdot (-\kappa(T) \nabla T) = 0$$

$$\epsilon(T) \sim T^2 \quad \kappa(T) \sim T \quad T = \partial_t F_r \quad \phi_a = -\frac{F_a}{\partial_t F_r}$$

- We can obtain all transport coefficients, including **stochastic transport coefficients**, at all orders.

Quantum energy fluctuations

- The SYK chain theory leads to **quantum corrections** in the SK EFT for energy diffusion, compatible with quantum KMS symmetry.

$$\mathcal{K} F_1(t) = -F_1(-t)$$

$$\mathcal{K} F_2(t) = -F_2(-t - i\hbar\beta) - i\hbar$$

- The structure of corrections is subtly different from charge diffusion

$$\partial_t n + \nabla \cdot \mathbf{J} = 0 \quad \longleftrightarrow \quad \text{U(1) symmetry (linear)}$$

$$\partial_t \epsilon + \nabla \cdot \mathbf{J}_\epsilon = 0 \quad \longleftrightarrow \quad \text{Time translation symmetry (non-linear)}$$

- **Quantum fluctuations** are controlled by the hierarchy

$$\omega \ll \frac{1}{t_{\text{micro}}} = \frac{\kappa/a^2}{\partial\epsilon/\partial T} \lesssim \frac{1}{\hbar\beta}$$

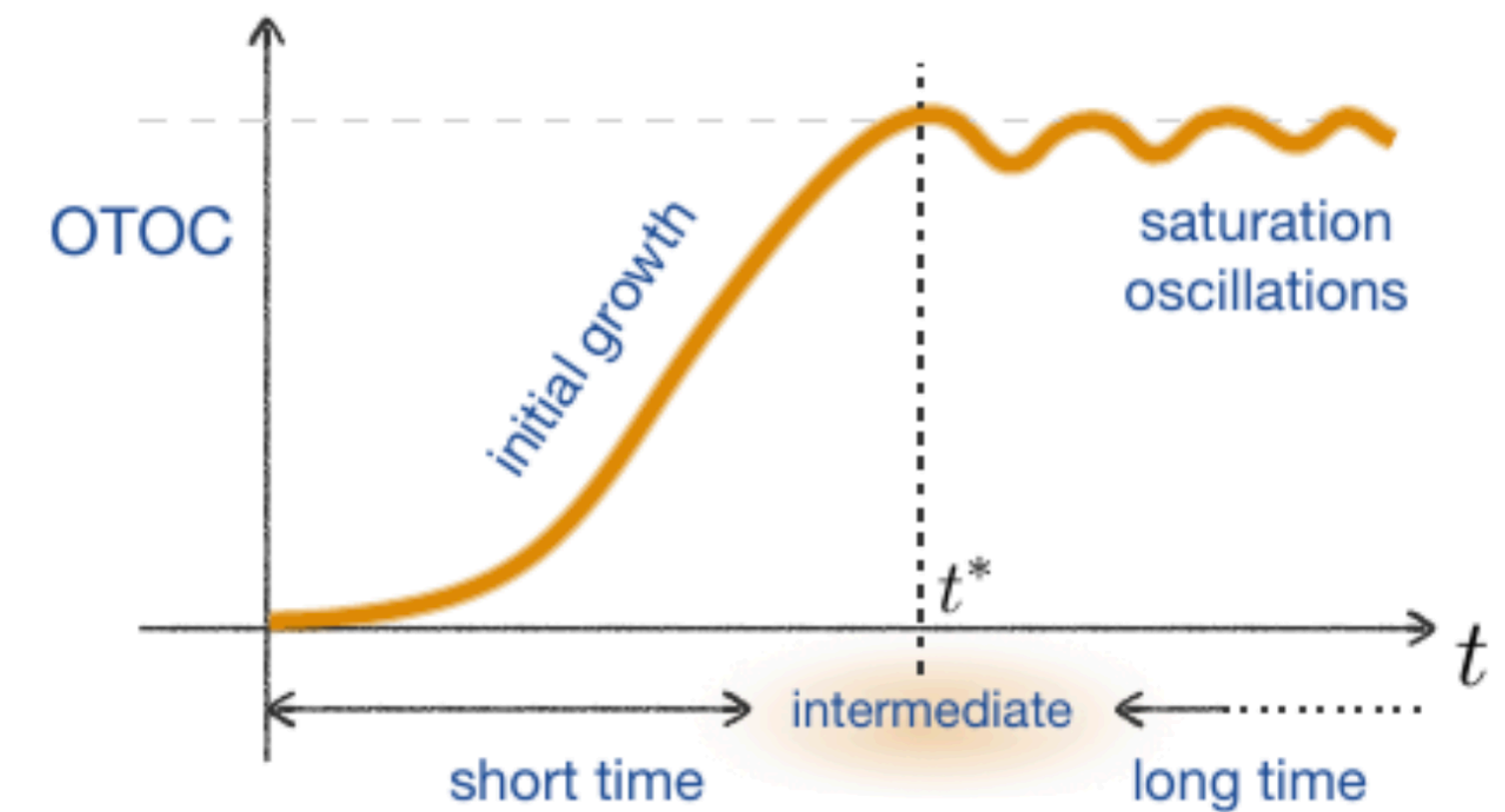
independent of temperature

Out-of-Time-Ordered Correlators (OTOCs)

- OTOCs serve as a probe of **chaos in quantum systems**

$$\left\langle [W(t), V(0)]^2 \right\rangle$$
$$\sim \exp(\lambda_L t) \quad (\text{early time})$$

Lyapunov exponent



[García-Mata et al. 2022]

- Scrambling admits a **maximal bound** [Maldacena, Shenker, Stanford 2022]

$$\lambda_L \leq \frac{2\pi}{\hbar\beta}$$

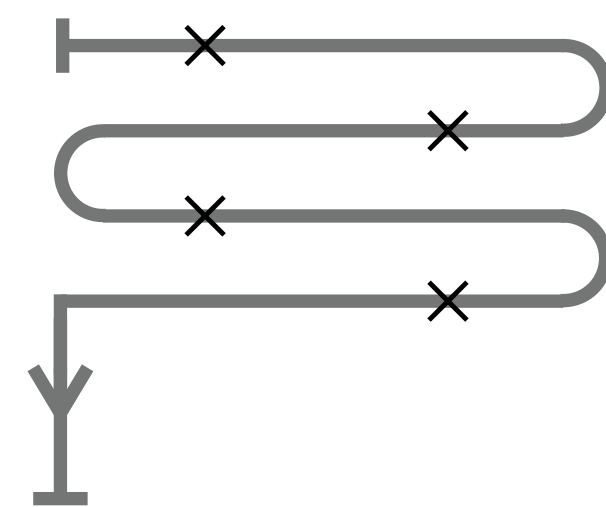
e.g. SYK, large- c CFTs

- EFTs for (nearly) maximal chaotic systems. [Blake, Lee, Liu 2018]
[Haehl, Rozali 2018]

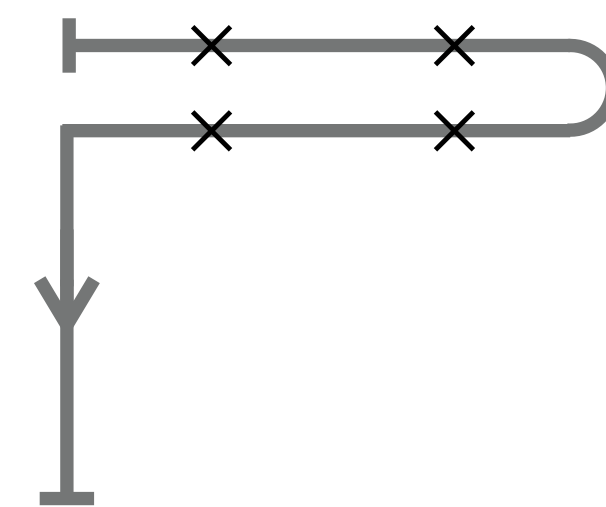
OTOCs and Quantum Chaos

- SK-EFTs for OTOCs require 4-contours.

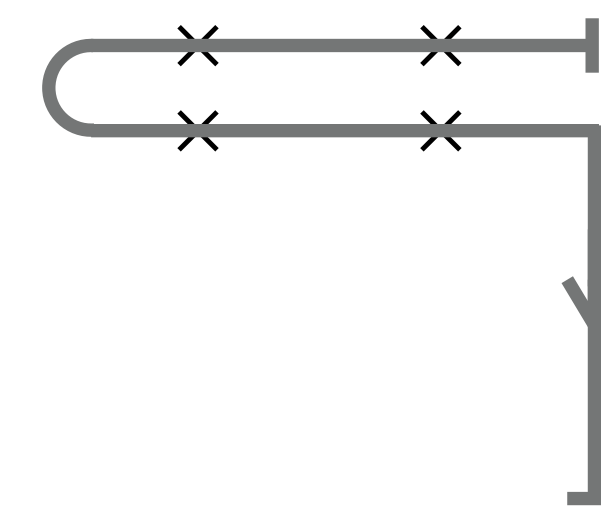
$$\begin{aligned} \langle [W(t), V(0)]^2 \rangle &= \langle W(t) V(0) W(t) V(0) \rangle - \langle V(0) W(t) W(t) V(0) \rangle \\ &\quad + \langle V(0) W(t) V(0) W(t) \rangle - \langle W(t) V(0) V(0) W(t) \rangle \end{aligned}$$



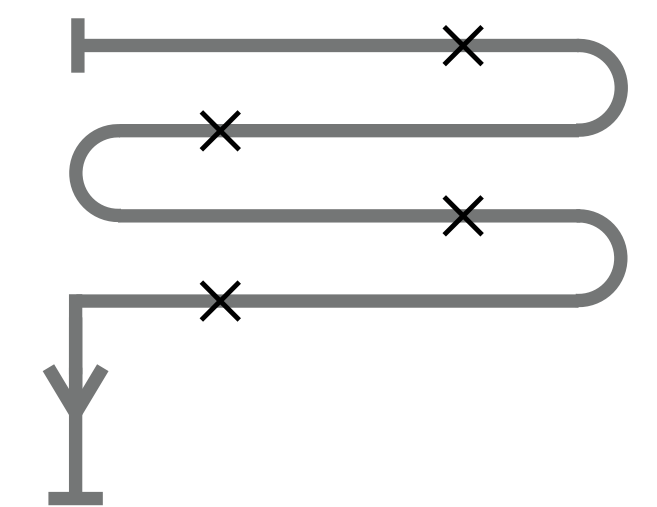
WWVW



-VWWV



-WVWV



VWVW

- We need to find appropriate unitarity conditions and KMS symmetry.

Schwinger-Keldysh Action on 4-Contour

- 4-Contour SK generating functional

$$\mathcal{Z}[A_{1,2,3,4}] = \int \mathcal{D}\phi_{1,2,3,4} \exp\left(i S[\phi_{1,2,3,4}; A_{1,2,3,4}]\right)$$

- Unitarity constraints

$$S_{kkij} = S_{ikkj} = S_{ijkk} = S_{ij}$$

$$S_{ijkl}^* = -S_{lkji}$$

$$\text{Im } S_{ijkl} \stackrel{?}{\geq} 0$$

$$F_i \equiv F[f_i]$$

$$F_{ij} \equiv F[f_i, f_j]$$

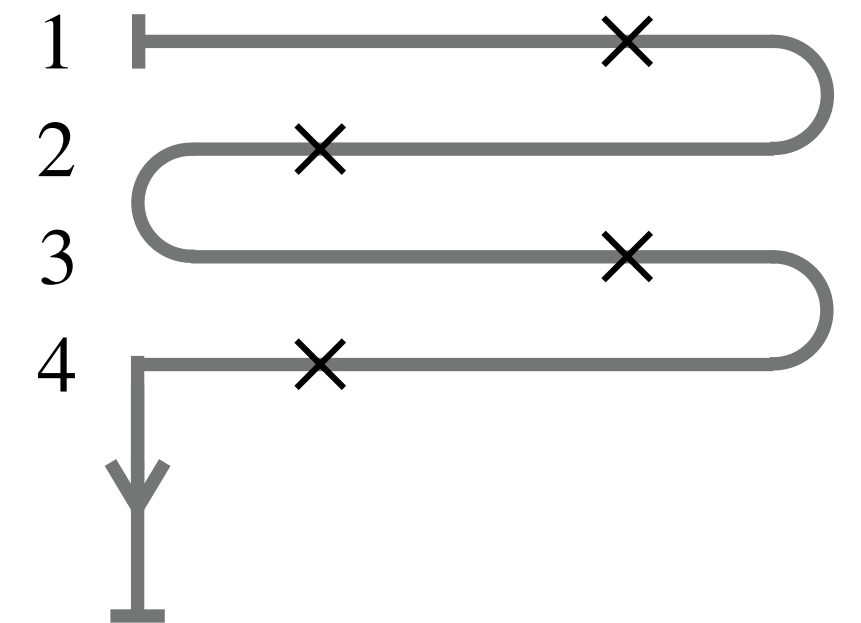
$$F_{ijkl} \equiv F[f_i, f_j, f_k, f_l]$$

$$S_{1234} = \underbrace{S_{12} + S_{23} + S_{34} + S_{14} - S_{13} - S_{24}}_{\text{Hydrostatic + Dissipative (Hydrodynamic)}} + \underbrace{\mathcal{A}_{1234}}_{\text{Chaotic}}$$

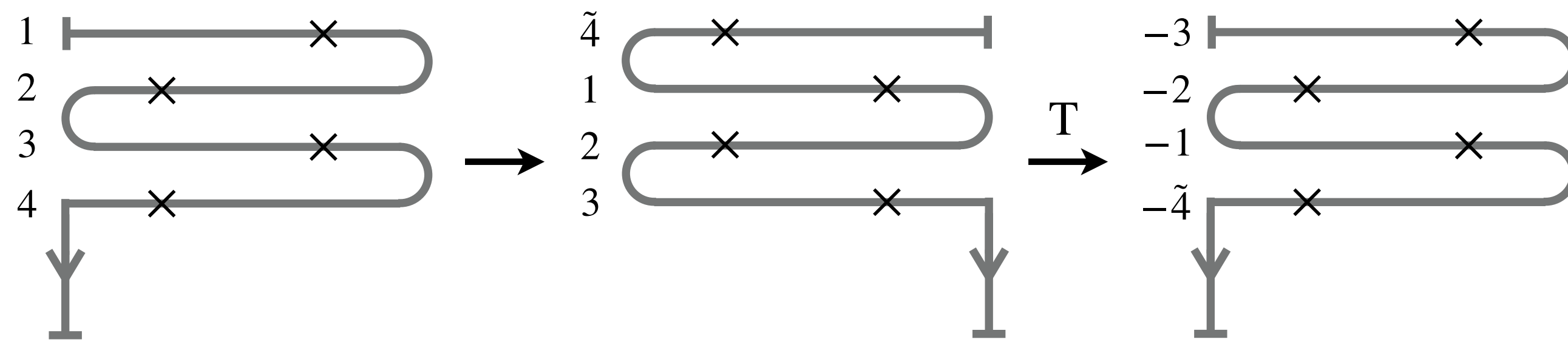
“4-Influence Functional”

Incompatible with KMS

$$\text{e.g. } \sim \cancel{(B_{2t} \ B_{3t})} \cancel{(\mathbf{B}_1 \ \mathbf{B}_2)} \cancel{(\mathbf{B}_3 \ \mathbf{B}_4)}$$



KMS Symmetry on a 4-Contour

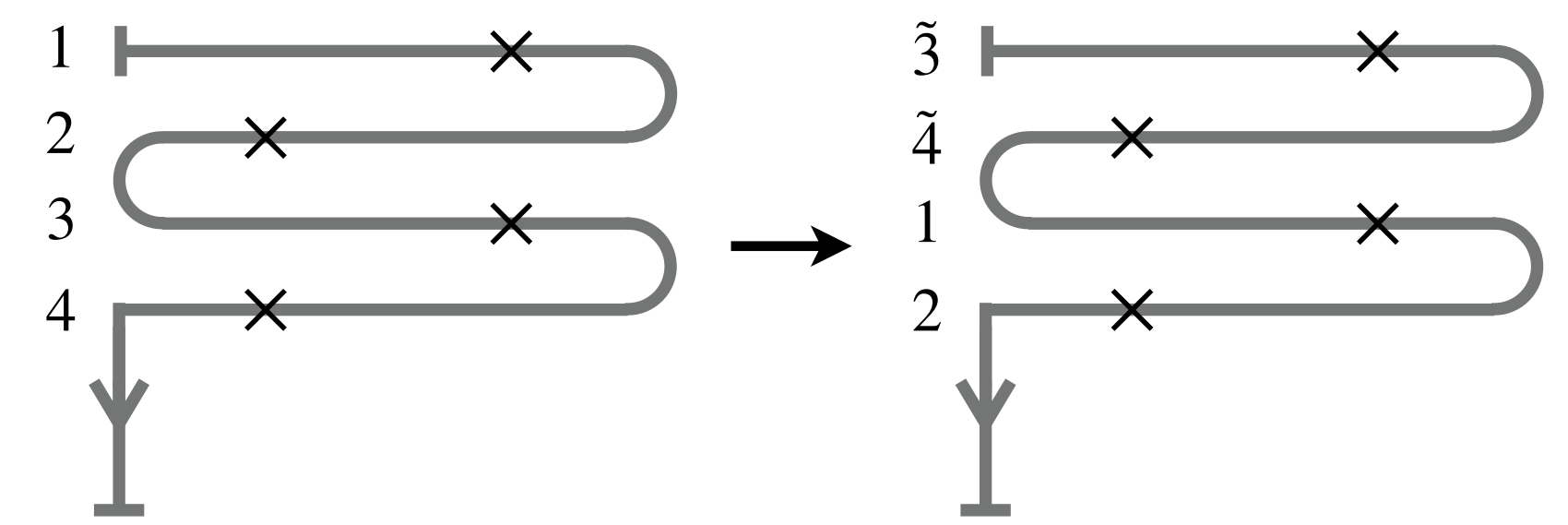


$$Kf_1(t) = \pm f_3(-t)$$

$$Kf_2(t) = \pm f_2(-t)$$

$$Kf_3(t) = \pm f_1(-t)$$

$$Kf_4(t) = \pm f_4(-t - i\hbar\beta)$$



$$Cf_1(t) = f_3(t - i\hbar\beta/2)$$

$$Cf_2(t) = f_4(t - i\hbar\beta/2)$$

$$Cf_3(t) = f_1(t + i\hbar\beta/2)$$

$$Cf_4(t) = f_2(t + i\hbar\beta/2)$$

$$K^2 = 1 \quad C^2 = 1 \quad KCK = C$$

→ Dih₂

Implies constraints on $\mathcal{A}_{1234}$

► Generalises to Dih_n for SK-EFTs on 2n-contour.



Outlook

- We have analysed quantum fluctuations in SK hydrodynamics.
- We focused on the simplest model of an Abelian diffusive charge.
TODO: generalise to conserved energy and momentum.
- We computed one-loop corrections to the retarded correlation functions arising from quantum fluctuations.
TODO: generalise to higher-loops where the computation becomes sensitive to new quantum interaction vertices.
- Physics question: are there any physical setups where we can cleanly observe/compute the quantum fluctuation signatures, distinct from the predictions from stochastic fluctuations?
Perhaps, SYK chains or holography?



Outlook

- Can we construct general 4-contour SK hydrodynamics compatible with the KMS symmetry?
How does the 4-influence functional affect correlation functions, especially OTOCs.
Is there a general prescription for higher-contours?
- Are there extra conditions for maximal chaos? $SL(2, \mathbb{R})$?
What about near-maximal chaos?
[Blake, Lee, Liu 2018]
[Haehl, Rozali 2018]
- Does the physics of $\mathcal{A}_{1234}$ survive in $\hbar \rightarrow 0$ limit?
If so, what is its stochastic interpretation?
- WIP: deriving the 4-contour SK effective action for the SYK chain.



THANK YOU

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