

Stochastic
Is ~~Schwinger-Keldysh~~ Hydrodynamics Compatible
with Thermodynamic Equilibrium?

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© Phil Smith "Stochastic 2"

[250x.xxxxx] AJ, Kovtun

[2011.03691] AJ, Kovtun, Ritz, Shukla

Foundations and Applications of Relativistic Hydrodynamics

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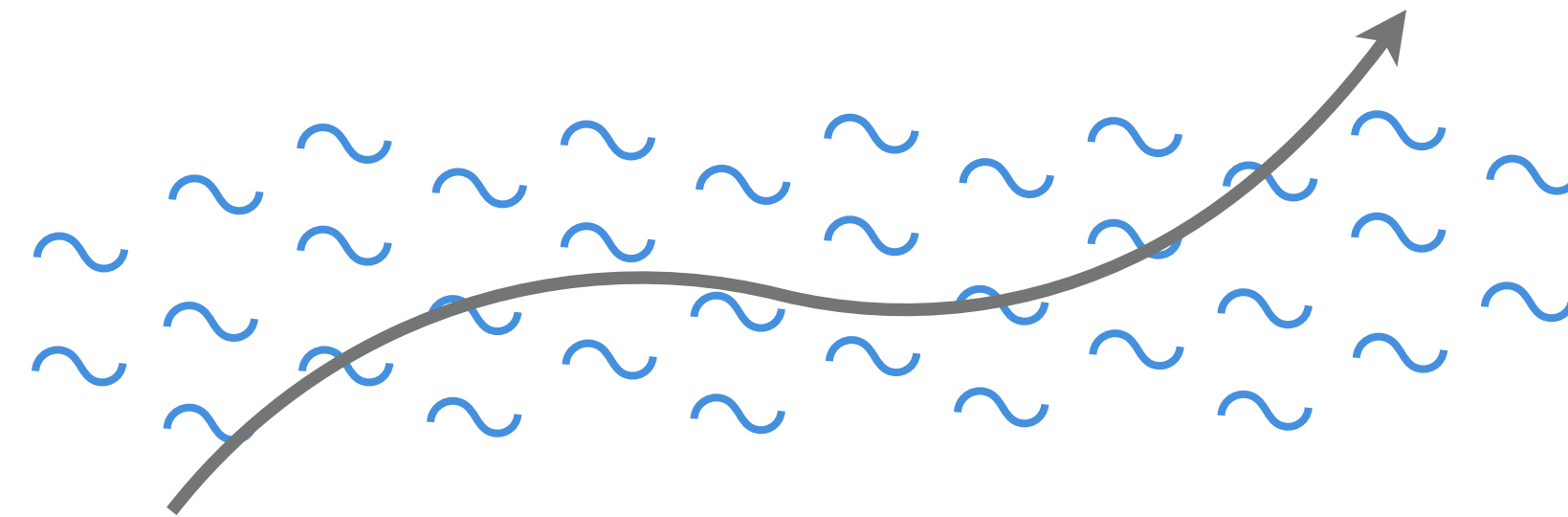
The big picture

- **Effective field theory (EFT)** is a powerful tool to construct low-energy effective models for complex many-body systems, without the detailed knowledge of their microscopic constituents.
- The conventional EFT framework is only suited to systems at zero temperature or systems in **thermal equilibrium** (statistical field theory).
- **Hydrodynamics** provides a universal framework to describe low-energy effective dynamics of complex many-body systems operating *close to thermal equilibrium*.
- Classical hydrodynamics is “*like*” an EFT. It involves *effective fields* for local fluid velocity, temperature, density, etc, whose dynamics is dictated by universal symmetry principles. However, it is *not* based on an **effective action principle**.

This has changed recently!

The big picture

- Classical hydrodynamics has limited applicability due to the presence of **stochastic thermal fluctuations**.



- Classical hydrodynamics *includes dissipation but not fluctuations*, which are necessitated by the **fluctuation-dissipation theorem (FDT)**.
- **Schwinger-Keldysh (SK) hydrodynamics** is an EFT extension of classical hydrodynamics, which systematically incorporates the effects of stochastic thermal fluctuations.

[Kovtun, Moore, Romatschke 2014]

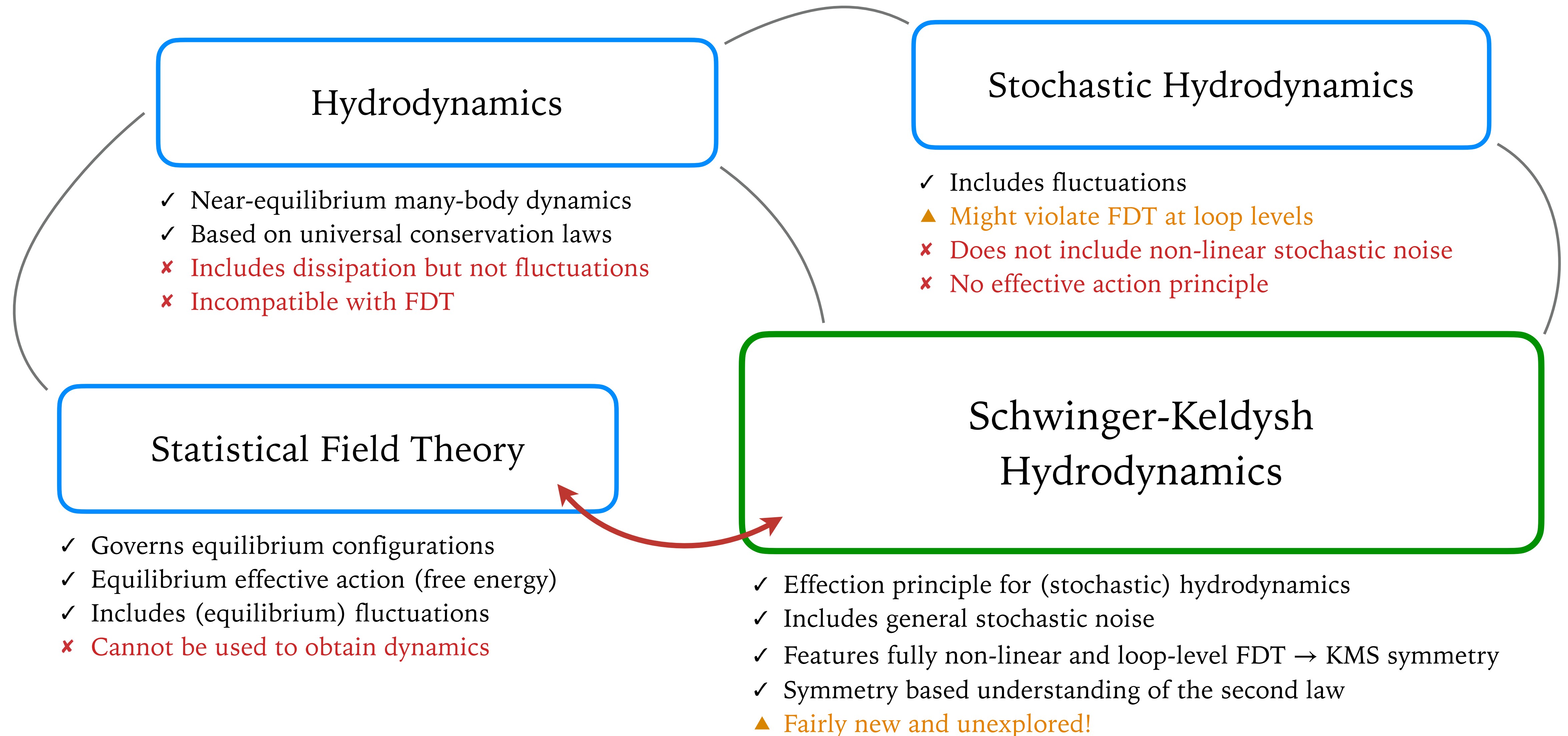
[Harder, Kovtun, Ritz 2015]

[Crossley, Glorioso, Liu 2015]

[Haehl, Loganayagam, Rangamani 2015]

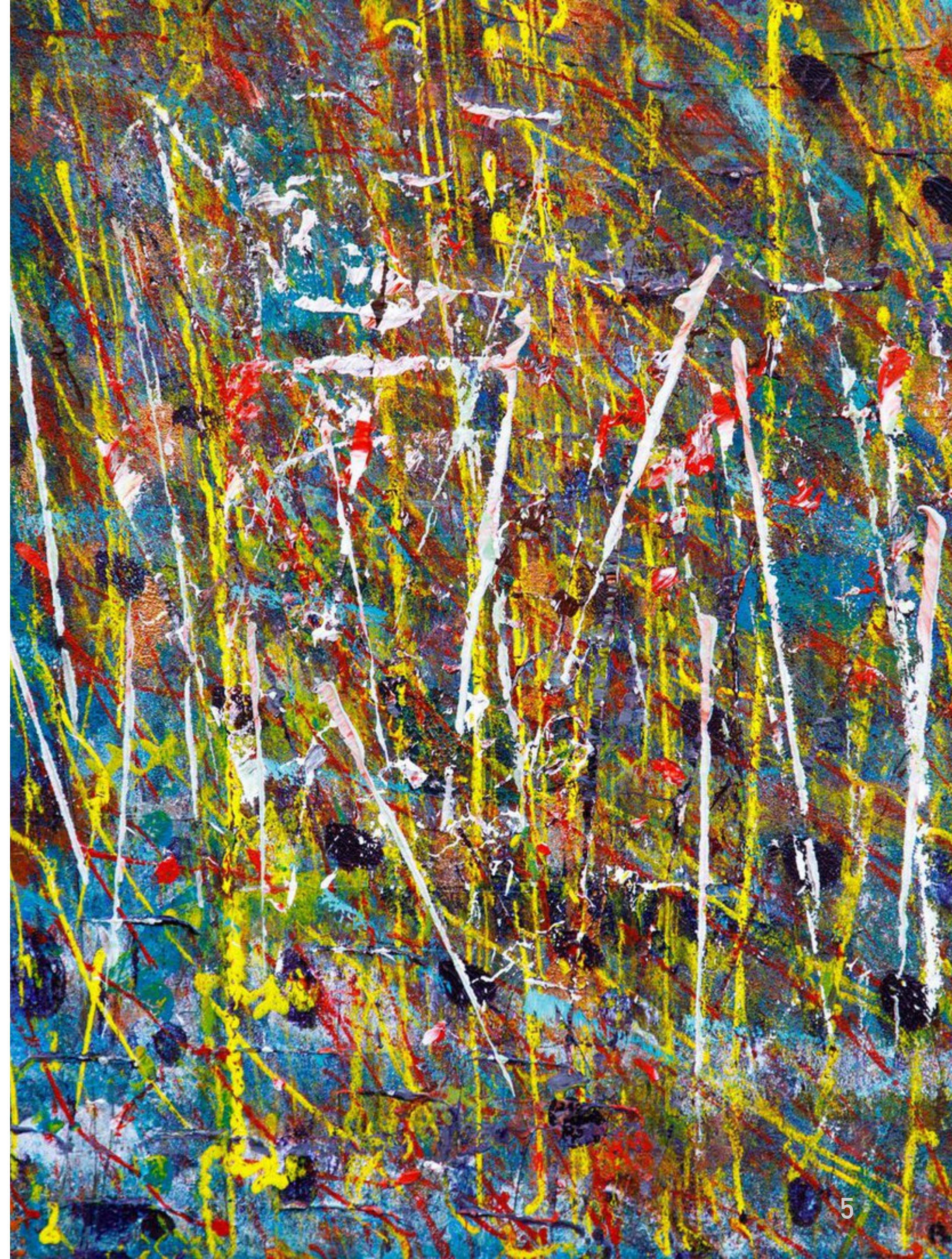
[Jensen, Pinzani-Fokeeva, Yarom 2017]

The hydrodynamic landscape



Outline

- **Hydrostatic compatibility**
- Schwinger-Keldysh hydrodynamics
- Hydrostatic compatibility at one-loop
- BRST ghosts
- Lessons and outlook



Hydrodynamics 101

- Consider hydrodynamics of a conserved scalar variable ψ , with **conservation equation**

$$\partial_t \psi + \nabla \cdot \mathbf{J} = 0$$

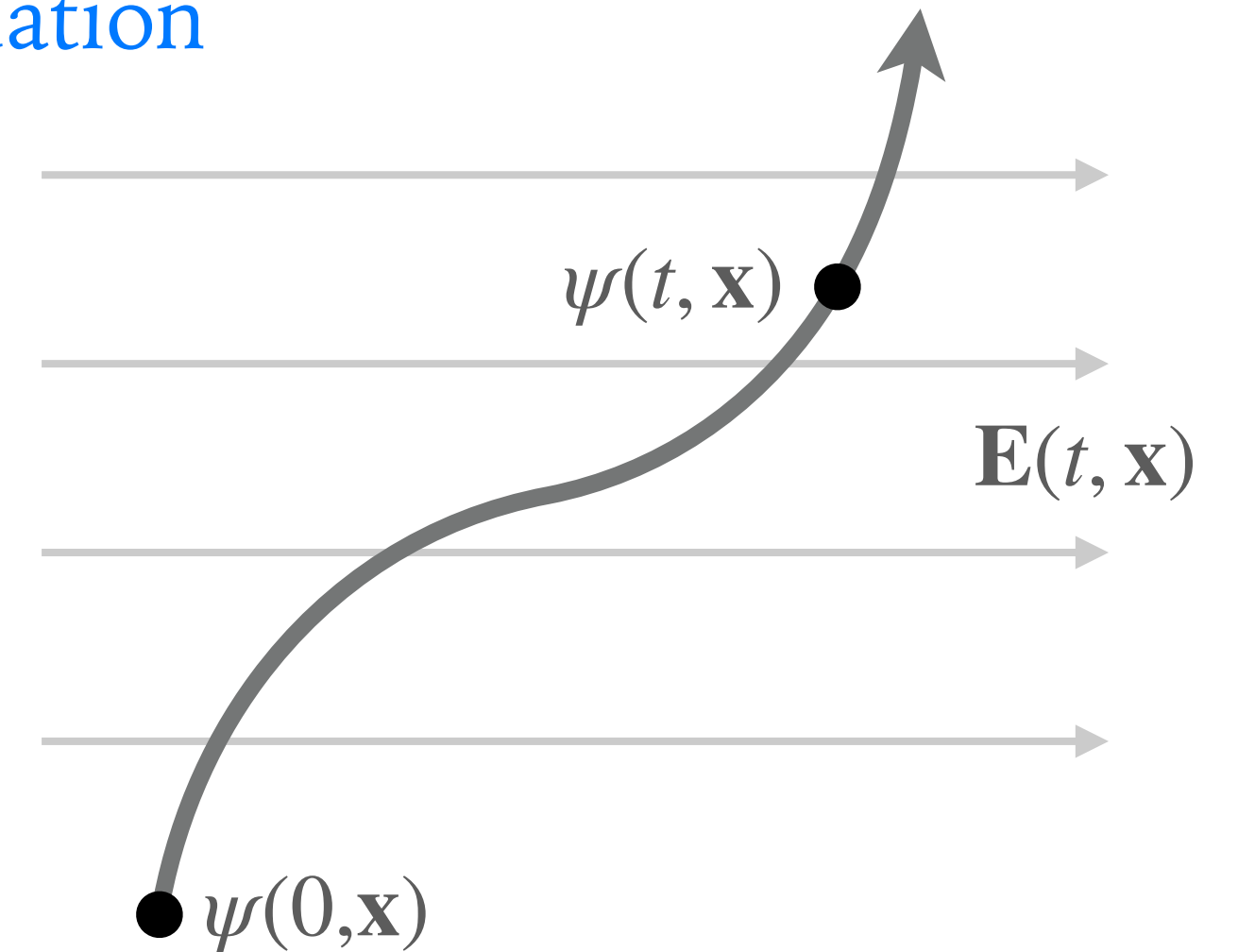
Given $\mathbf{J} = \mathbf{J}(\psi)$, this equation governs the classical evolution of ψ .

- The **constitutive relations** are given as

$$\mathbf{J} = -D(\psi) \nabla \psi + \sigma(\psi) \mathbf{E} + \mathcal{O}(\nabla^2)$$

↓
diffusion

↓
electric
conductivity



$\mathbf{E} = \nabla h$ is applied electric field

- The diffusion and conductivity are related via the chemical potential $\mu(\psi) = \delta F[\psi]/\delta \psi$,

$$D(\psi) = \mu'(\psi) \sigma(\psi)$$

$$\implies \mathbf{J} = -\sigma(\psi) \left(\nabla \mu(\psi) - \mathbf{E} \right) + \mathcal{O}(\nabla^2)$$

Arises from the local second law of thermodynamics, together with $\sigma \geq 0$.

Hydrostatic compatibility

- The **second law equality constraints**, such as the diffusion-conductivity constraint, ensure the **hydrostatic compatibility** of classical hydrodynamics:

In the presence of time-independent background sources, hydrodynamic equations must admit a time-independent equilibrium solution, governed by an **equilibrium generating functional**.

[Banerjee et al. 2012]

[Jensen et al. 2012]

$$\mathcal{Z}_{\text{eqb}}[h] = \int_{\text{eqb}} \mathcal{D}\psi \exp\left(-F[\psi] + \int_{\mathbf{x}} h \psi\right)$$

- Classically, the generating functional is dominated by the saddle point $\mu(\psi_{\text{eqb}}) = h$. This solves the hydrodynamic equation, provided that the **second law equality constraints** are obeyed

$$\begin{aligned} \partial_t \psi_{\text{eqb}} + \nabla \cdot \mathbf{J}_{\text{eqb}} &= -\nabla \cdot \left(D(\psi_{\text{eqb}}) \nabla \psi_{\text{eqb}} - \sigma(\psi_{\text{eqb}}) \mathbf{E} \right) + \mathcal{O}(\nabla^3) \\ &= -\nabla \cdot \left(\left(\frac{D(\psi_{\text{eqb}})}{\mu'(\psi_{\text{eqb}})} - \sigma(\psi_{\text{eqb}}) \right) \nabla h \right) + \mathcal{O}(\nabla^3) = 0 \end{aligned}$$

Model A and Model B

- We will consider a slight generalisation of our scalar hydrodynamic model [Hohenberg, Halperin 1977]

$$\partial_t \psi + \Gamma \left(\mu(\psi) - h \right) = 0$$

$$\Gamma = \Gamma(\psi) \quad \text{if } \psi \text{ is not conserved (Model A)}$$

$$\Gamma = - \nabla \cdot (\sigma(\psi) \nabla \circ) \quad \text{if } \psi \text{ is conserved (Model B)}$$

Technically, only Model B is hydrodynamic.

- For concreteness, we use

$$F[\psi] = \frac{1}{2} \int_{\mathbf{x}} \left(\gamma (\nabla \psi)^2 + a_2 \psi^2 + \frac{1}{3} a_3 \psi^3 + \frac{1}{12} a_4 \psi^4 \right)$$

$$\mu(\psi) = (a_2 - \gamma \nabla^2) \psi + \frac{1}{2} a_3 \psi^2 + \frac{1}{6} a_4 \psi^3$$

$$\Gamma(\psi) = \Gamma_0 + \Gamma_1 \psi + \frac{1}{2} \Gamma_2 \psi^2 \quad \text{(Model A)}$$

$$\sigma(\psi) = \sigma_0 + \sigma_1 \psi + \frac{1}{2} \sigma_2 \psi^2 \quad \text{(Model B)}$$

Hydrostatic compatibility of correlation functions

- Hydrodynamics can be used to compute *classical retarded correlation functions*

[Kadanoff, Martin 1963]

$$G^{\text{ret,cl}} \equiv \frac{\delta}{\delta h} \psi_{\text{onshell}}[h]$$

$$\partial_t \psi + \Gamma(\mu - h) = 0$$

$$\mu = -\gamma \nabla^2 \psi + a_2 \psi + \dots$$

$$G^{\text{ret,cl}}(\omega, \mathbf{k})_{(\text{A})} = \frac{\Gamma_0}{(a_2 + \gamma \mathbf{k}^2) \Gamma_0 - i\omega} \quad G^{\text{ret,cl}}(\omega, \mathbf{k})_{(\text{B})} = \frac{\sigma_0 \mathbf{k}^2}{(a_2 + \gamma \mathbf{k}^2) \sigma_0 \mathbf{k}^2 - i\omega}$$

$$\Gamma_{(\text{A})} = \Gamma_0 + \dots$$

$$\Gamma_{(\text{B})} = -\nabla((\sigma_0 + \dots) \nabla \circ)$$

- The *equilibrium correlation functions* are governed by the generating functional

$$G^{\text{eqb}} = \frac{\delta}{\delta h} \frac{\delta}{\delta h} \ln \mathcal{Z}_{\text{eqb}}[h] \longrightarrow G^{\text{eqb,cl}} = \frac{\delta}{\delta h} \psi_{\text{eqb}}[h]$$

$$\mathcal{Z}_{\text{eqb}}[h] = \int_{\text{eqb}} \mathcal{D}\psi \exp\left(-F[\psi] + \int_{\mathbf{x}} h \psi\right)$$

$$G^{\text{eqb,cl}}(\mathbf{k}) = \frac{1}{a_2 + \gamma \mathbf{k}^2}$$

$$F[\psi] = \frac{1}{2} \int_{\mathbf{x}} \left(\gamma (\nabla \psi)^2 + a_2 \psi^2 + \dots \right)$$

- Hydrostatic compatibility implies

$$G^{\text{ret,cl}}(\omega = 0, \mathbf{k}) = G^{\text{eqb,cl}}(\mathbf{k})$$

Similar compatibility conditions exist for higher-point correlation functions.

Stochastic hydrostatic compatibility

- **Hydrostatic compatibility** must remain intact in the presence of interactions

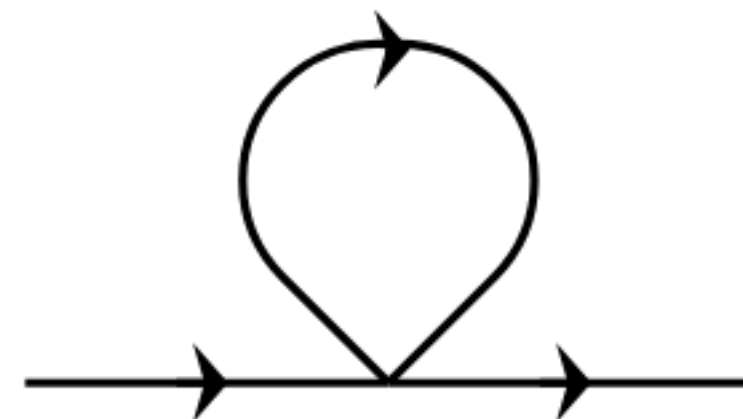
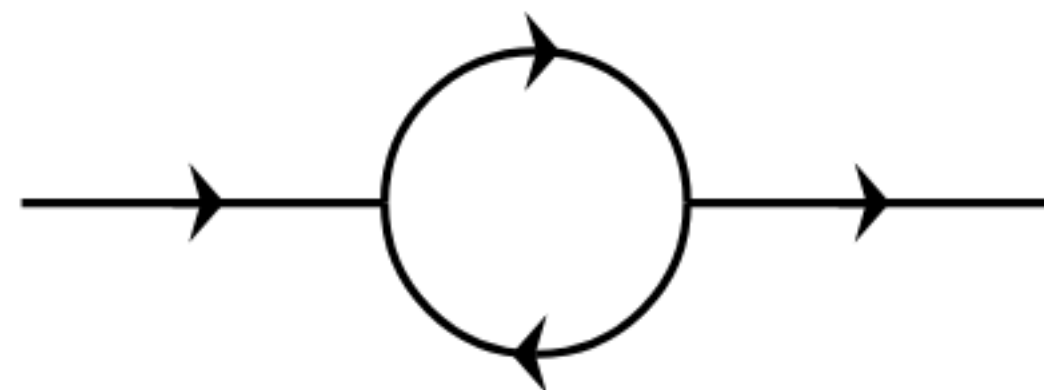
$$G^{\text{ret,cl}}(\omega = 0, \mathbf{k}) = G^{\text{eqb,cl}}(\mathbf{k})$$



$$G^{\text{ret}}(\omega = 0, \mathbf{k}) \stackrel{?}{=} G^{\text{eqb}}(\mathbf{k})$$

- $G^{\text{eqb}}(\mathbf{k})$ can be obtained using the equilibrium generating functional.

One-loop corrections:



$$F[\psi] = \frac{1}{2} \int_{\mathbf{x}} \left(\gamma (\nabla \psi)^2 + a_2 \psi^2 + \frac{1}{3} a_3 \psi^3 + \frac{1}{12} a_4 \psi^4 \right)$$

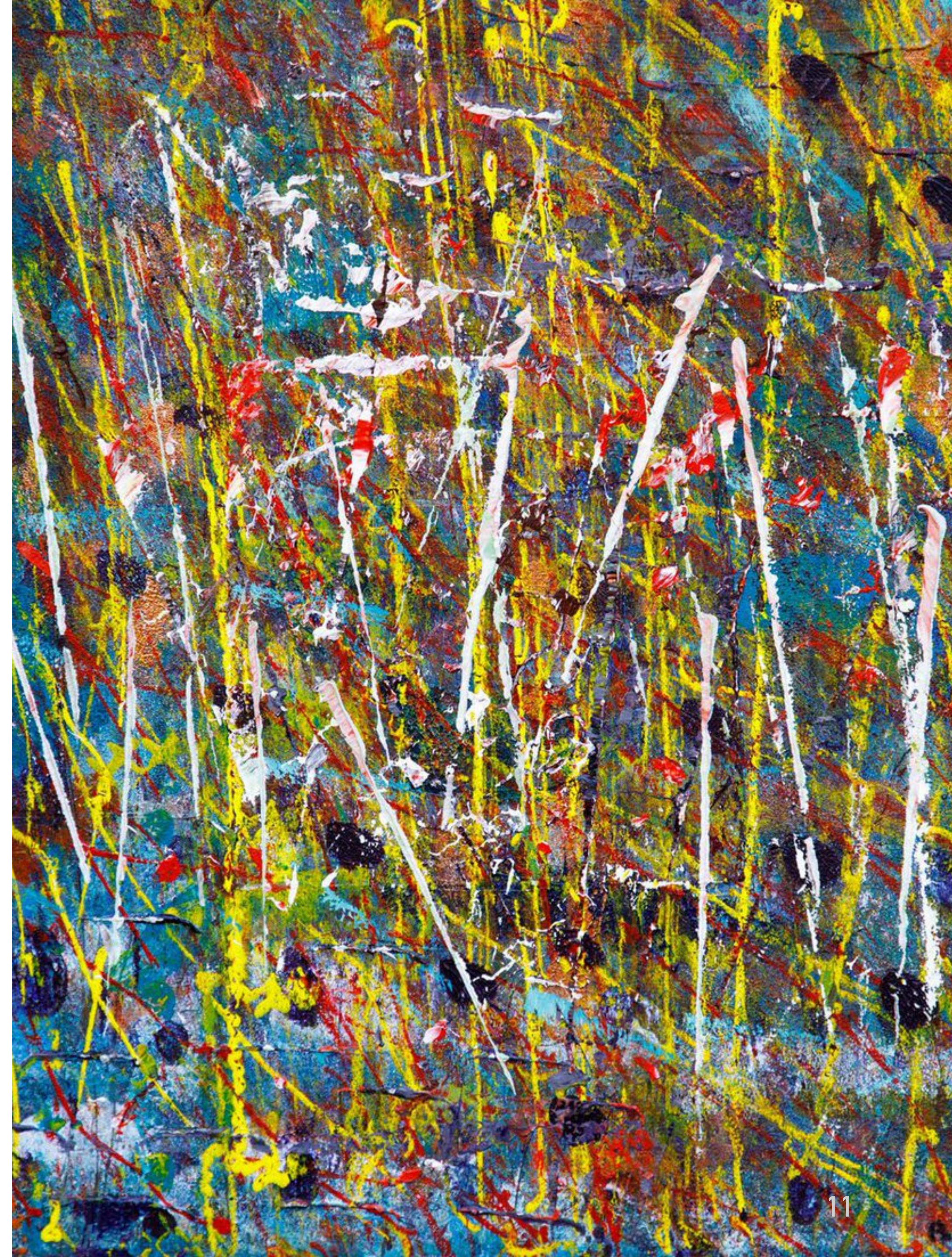
- To compute $G^{\text{ret}}(\mathbf{k})$, we need the framework of **Schwinger-Keldysh (stochastic) hydrodynamics**.

Is Schwinger-Keldysh hydrodynamics
compatible with hydrostatics?

Depends...

Outline

- Hydrostatic compatibility
- **Schwinger-Keldysh hydrodynamics**
- Hydrostatic compatibility at one-loop
- BRST ghosts
- Lessons and outlook



Stochastic fluctuations

- **Fluctuation-dissipation theorem** relates retarded correlation functions to **symmetric correlation functions**

$$\langle \psi^2 \rangle - \langle \psi \rangle^2 \quad G^{\text{sym}}(\omega, \mathbf{k}) = \frac{2}{\omega} \text{Im} G^{\text{ret}}(\omega, \mathbf{k}) \quad \psi(t) \sim e^{-\#t}$$

- Symmetric functions compute **stochastic fluctuations** of a variable, which are absent in classical hydrodynamics.
- Fluctuations *backreact* on hydrodynamics via interactions and induce **stochastic-loop corrections** in retarded correlation functions.
- **Schwinger-Keldysh (SK) hydrodynamics** provides a systematic symmetry-based EFT framework that incorporates the effects of stochastic thermal fluctuations.

*We will take $\beta = 1/(k_B T) = 1$.

MSR path integral

- Stochastic hydrodynamics with **Gaussian noise**:

$$\langle \theta(t, \mathbf{x}) \rangle = 0$$

$$\langle \theta(t, \mathbf{x}) \theta(t', \mathbf{x}') \rangle = 2 \delta(t - t', \mathbf{x} - \mathbf{x}')$$

$$\underbrace{\partial_t \psi_r + \Gamma \left(\mu(\psi_r) - h_r \right)}_E + f(\psi_r) \theta = 0 \quad f^2 = \Gamma \text{ due to FDT}$$

$$\langle \dots \rangle = \int \mathcal{D}\theta (\dots)_{\text{onshell}} \exp \left(-\frac{1}{4} \int_x \theta^2 \right) \quad \begin{array}{l} \psi \equiv \langle \psi_r \rangle \\ h \equiv h_r \end{array}$$

- Martin-Siggia-Rose (MSR) path integral [Martin, Siggia, Rose 1973]

$$\mathcal{Z}[h_r, h_a] = \int \mathcal{D}\theta \mathcal{D}\psi_r \delta[E + f\theta] \exp \left(i \int_x \frac{i}{4} \theta^2 + h_a \psi_r \right) = \int \mathcal{D}\theta \mathcal{D}\psi_r \mathcal{D}\phi_a \exp \left(i \int_x \frac{i}{4} \theta^2 - \phi_a (E + f\theta) + h_a \psi_r \right)$$

$$G^{\text{ret}} = \frac{\delta}{\delta h_r} \frac{-i\delta}{\delta h_a} \ln \mathcal{Z}[h_r, h_a] = \frac{\delta}{\delta h_r} \langle \psi_r \rangle$$

$$= \int \mathcal{D}\psi_r \mathcal{D}\phi_a \exp \left(\underbrace{i \int_x i\Gamma \phi_a^2 - \phi_a E + h_a \psi_r}_{S[\psi_r, \phi_a; h_r, h_a]} \right)$$

$$G^{\text{sym}} = \frac{-i\delta}{\delta h_a} \frac{-i\delta}{\delta h_a} \ln \mathcal{Z}[h_r, h_a] = \langle \psi_r^2 \rangle - \langle \psi_r \rangle^2$$

Schwinger-Keldysh generating functional

- Schwinger-Keldysh generating functional $\mathcal{Z}[h_r, h_a]$

$$G^{\text{ret}} = \frac{\delta}{\delta h_r} \frac{-i\delta}{\delta h_a} \ln \mathcal{Z}[h_r, h_a] \quad G^{\text{sym}} = \frac{-i\delta}{\delta h_a} \frac{-i\delta}{\delta h_a} \ln \mathcal{Z}[h_r, h_a]$$

h_r : **physical** background field

h_a : **noise** background field

(same as the classical background field h)

- Microscopic unitarity constraints:

$$\mathcal{Z} \Big|_{h_a \rightarrow 0} = 1 \quad \mathcal{Z} \Big|_{h_a \rightarrow -h_a} = \mathcal{Z}^* \quad \text{Re } \mathcal{Z} \leq 0 \quad \Rightarrow$$

Poles of $G^{\text{ret}}(\omega, \mathbf{k})$ lie at $\text{Im } \omega < 0$

$$G^{\text{ret}}(t - t', \mathbf{x} - \mathbf{x}') \propto \Theta(t - t')$$

$$G^{\text{sym}}(\omega, \mathbf{k}) \geq 0$$

- Kubo-Martin-Schwinger (KMS) symmetry:

$$\text{K } h_r(t) = h_r(-t)$$

$$\text{K } h_a(t) = \left(h_a + i\partial_t h_r \right)(-t)$$

$\Rightarrow$

$$G^{\text{sym}}(\omega, \mathbf{k}) = \frac{2}{\omega} \text{Im } G^{\text{ret}}(\omega, \mathbf{k})$$

Fluctuation-Dissipation Theorem

Schwinger-Keldysh effective field theory

- Schwinger-Keldysh effective field theory (SK-EFT)

$$\mathcal{Z}[h_r, h_a] = \int \mathcal{D}\Psi_r \mathcal{D}\Psi_a \exp\left(i S[\Psi_r, \Psi_a; h_r, h_a]\right)$$

Ψ_r : **physical** dynamical field(s)

Ψ_a : **noise** dynamical field(s)

(not necessarily the classical field ψ)

- The SK-EFT must correctly reproduce the classical hydrodynamic equations.

- The SK-EFT must satisfy:

- ✓ Unitarity constraints:

$$\mathcal{Z} \Big|_{h_a \rightarrow 0} = 1$$



$$S \Big|_{h_a, \Psi_a \rightarrow 0} = 0$$

$$\mathcal{Z} \Big|_{h_a \rightarrow -h_a} = \mathcal{Z}^*$$



$$S \Big|_{h_a \rightarrow -h_a, \Psi_a \rightarrow -\Psi_a} = -S^*$$

$$\text{Re } \mathcal{Z} \leq 0$$



$$\text{Im } S \geq 0$$

- ✓ KMS symmetry:

$$\mathbf{K} h_r(t) = h_r(-t)$$

$$\mathbf{K} h_a(t) = \left(h_a + i\partial_t h_r\right)(-t)$$

$$\mathbf{K} \Psi_r(t) = \pm \Psi_r(-t)$$

$$\mathbf{K} \Psi_a(t) = \pm \left(\Psi_a + i\partial_t \Psi_r\right)(-t)$$

time-reversal
eigenvalue of $\Psi_{r,a}$

Schwinger-Keldysh hydrodynamics 101

- There may be **multiple SK-EFTs** for a hydrodynamic model. Consider $(S = \int dt dx \mathcal{L})$ $\partial_t \psi + \Gamma(\mu - h) = 0$

Canonical Scheme (Can)

$$\mathcal{L} = -\psi_a \left(\mu(\psi_r) - h_r \right) + \psi_a \frac{i}{\Gamma} \left(\psi_a + i\partial_t \psi_r \right) + h_a \psi_r$$

$$K \psi_r(t) = \psi_r(-t)$$

$$K \psi_a(t) = \left(\psi_a + i\partial_t \psi_r \right)(-t)$$

Not suitable for Model B

Grand Canonical Scheme (GCan)

$$\mathcal{L} = \psi(\partial_t \phi_r + h_r) \left(\partial_t \phi_a + h_a \right) + i\phi_a \Gamma \left(\phi_a + i\partial_t \phi_r \right)$$

$$K \phi_r(t) = -\phi_r(-t)$$

$$K \phi_a(t) = -\left(\phi_a + i\partial_t \phi_r \right)(-t)$$

$$\psi_a \leftrightarrow \Gamma \phi_a$$

$$\mu(\psi_r) \leftrightarrow \partial_t \phi_r + h_r$$

$$\psi \equiv \psi_r$$

$$\mu(\psi) \equiv \partial_t \phi_r + h_r$$

- The KMS symmetries in the two schemes are physically distinct.*
- Field redefinitions affect the renormalisation structure of physical observables.

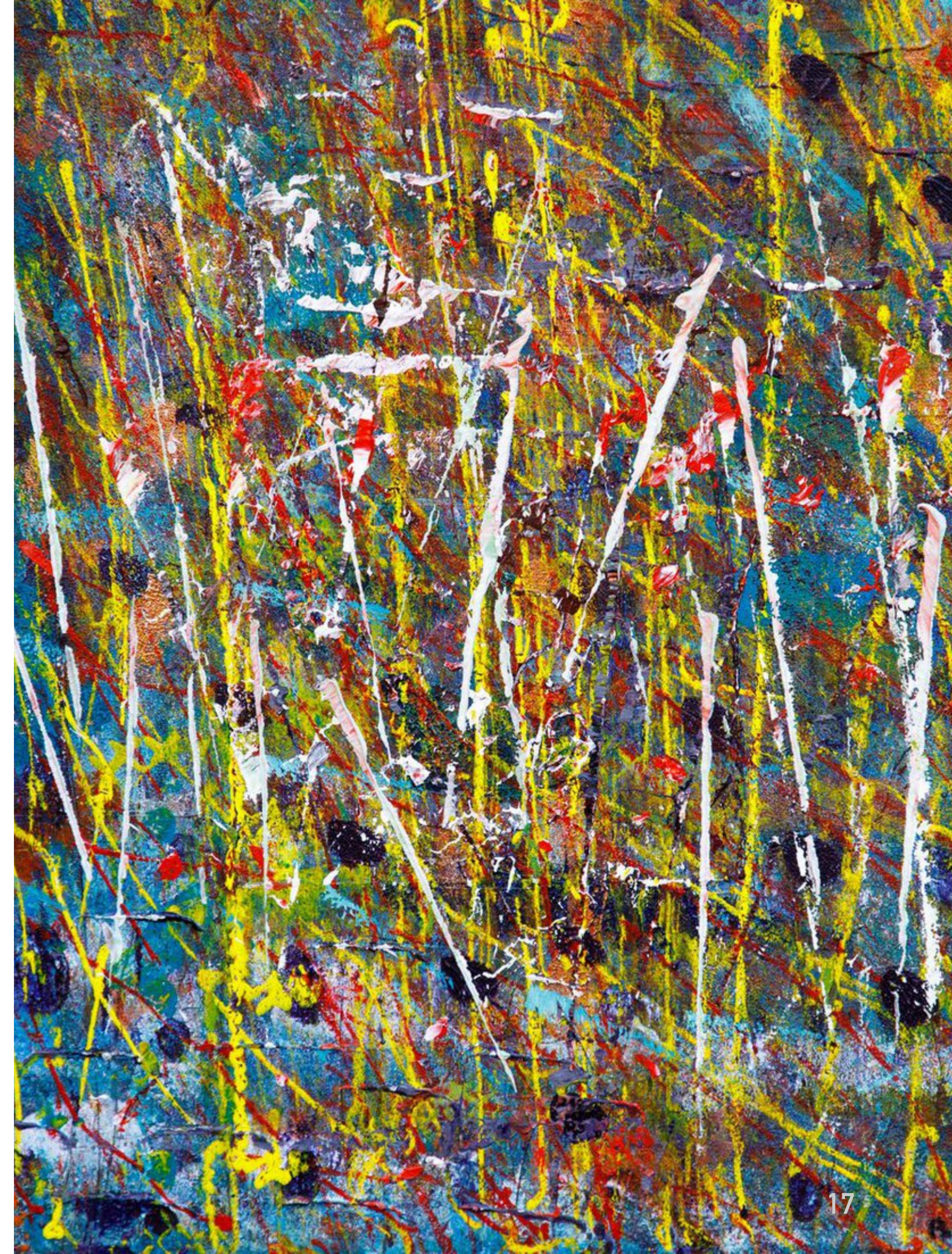
⇒

One or both SK-EFT schemes may be incompatible with hydrostatics due to interactions.

*In fact, $K_C \psi_r = K_{GC} \psi_r$, but $K_C \psi_a \stackrel{\text{onshell}}{=} -K_{GC} \psi_a$.

Outline

- Hydrostatic compatibility
- Schwinger-Keldysh hydrodynamics
- **Hydrostatic compatibility at one-loop**
- BRST ghosts
- Lessons and outlook



Stochastic Feynman rules

- Despite thier differences, the two SK-EFT schemes are qualitatively similar.
- Free propagators:

$$\text{wavy line with arrow} = 0$$

$$\text{black line} \rightarrow \text{wavy line} \sim \frac{1}{\Omega - i\omega}$$

Pole(s) only at $\text{Im } \omega < 0$

$$\text{black line} \rightarrow \text{black line} \sim \frac{1}{(\Omega - i\omega)(\Omega + i\omega)}$$

Pole(s) at both $\text{Im } \omega < 0$ and $\text{Im } \omega > 0$

$$\begin{array}{ll} \text{black line} & \Psi_r \\ \text{wavy line} & \Psi_a \end{array}$$

$$\Omega_{(A)} \equiv (a_2 + \gamma \mathbf{k}^2) \Gamma_0$$

$$\Omega_{(B)} \equiv (a_2 + \gamma \mathbf{k}^2) \sigma_0 \mathbf{k}^2$$

- Interactions:*

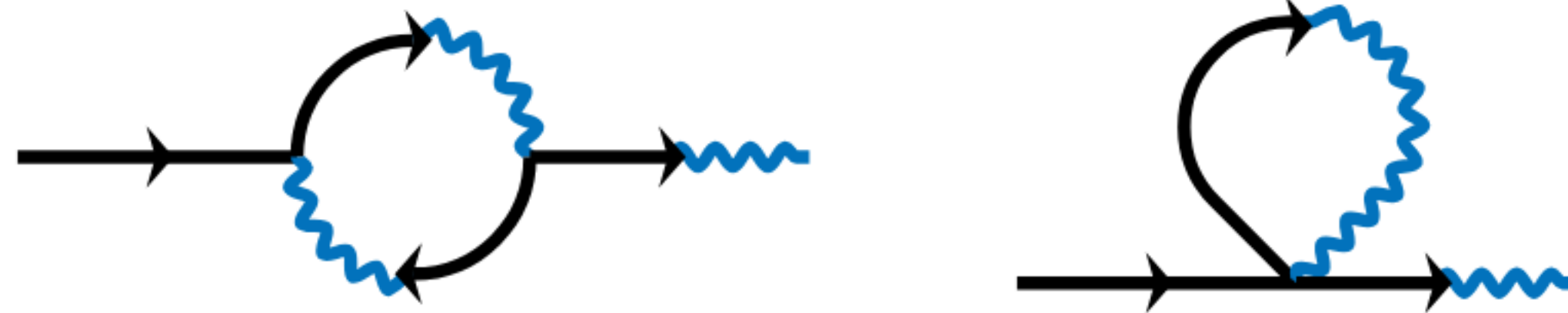


- These reproduce the classical retarded and symmetric correlation functions at tree-level.

*There are several other interactions involving background field insertions, which we do not explicitly mention here.

Non-unitary diagrams and GGL prescription

- The non-unitary diagrams:



Poles of $G^{\text{ret}}(\omega, \mathbf{k})$ ~~lie at $\text{Im } \omega < 0$~~

$$G^{\text{ret}}(t - t', \mathbf{x} - \mathbf{x}') \not\propto \Theta(t - t')$$

- What went wrong? Requiring $S|_{h_a, \Psi_a \rightarrow 0} = 0 \not\Rightarrow \mathcal{L}|_{h_a \rightarrow 0} = 0$ beyond tree-level.
- Quick fix: Regularise the retarded propagator ([GGL prescription](#)) [Gao, Glorioso, Liu 2018]

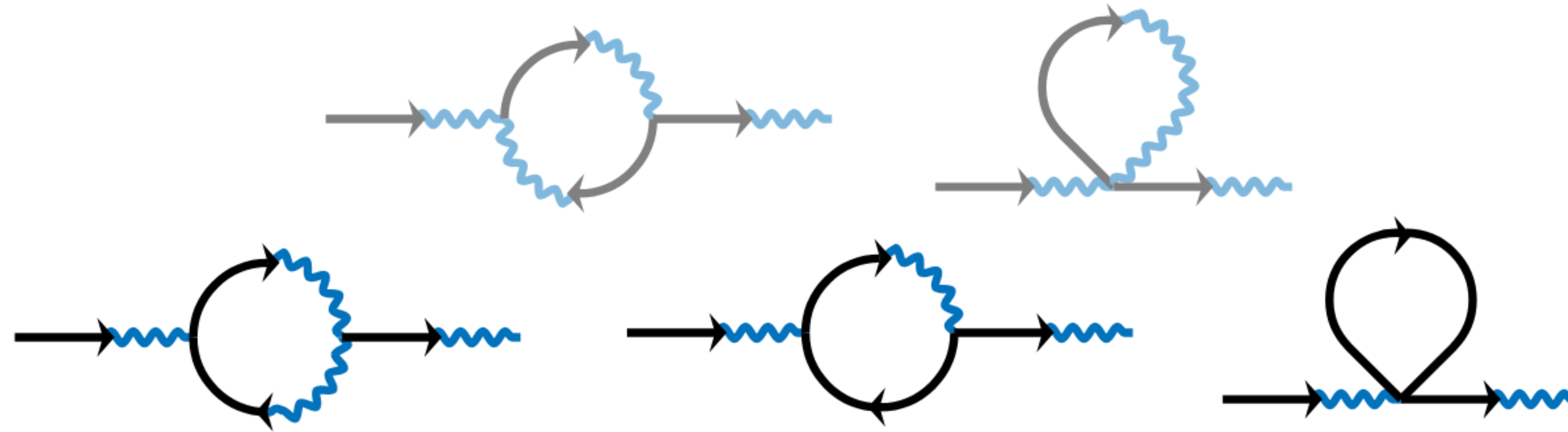
$$\longrightarrow \sim \frac{1}{\Omega - i\omega (1 - i\omega/M)^p} \iff (1 + 1/M \partial_t)^p \partial_t \psi + \Gamma (\mu(\psi) - h) = 0$$

$$\implies \text{retarded loops vanish}$$

- Caveat: $\text{Im } S \not\geq 0$: $i\psi_a \frac{1}{\Gamma} \left(1 - \frac{p(p-1)}{2M^2} \omega^2 + \dots \right) \psi_a \quad i\phi_a \Gamma \left(1 - \frac{p}{\Omega M} \omega^2 + \dots \right) \phi_a$
(dictated by KMS)

$$G^{\text{sym}}(\omega, \mathbf{k}) \not\geq 0$$

One-loop results and long-time tails



$$\mu(\psi) = (a_2 - \gamma \nabla^2) \psi + \frac{1}{2} a_3 \psi^2 + \frac{1}{6} a_4 \psi^3$$

$$\Gamma(\psi) = \Gamma_0 + \Gamma_1 \psi + \frac{1}{2} \Gamma_2 \psi^2$$

$$\sigma(\psi) = \sigma_0 + \sigma_1 \psi + \frac{1}{2} \sigma_2 \psi^2$$

$$\Omega \equiv a_2 \Gamma_0$$

$$D \equiv a_2 \sigma_0$$

- Cutoff-independent non-analytic contributions ($\gamma \rightarrow 0, d = 3$):

$$G^{\text{ret}}(\omega, \mathbf{k})_{(A)} = \frac{\Gamma_0}{\Omega - i\omega} + \mathcal{O}(\Lambda^3)$$

$$G^{\text{ret}}(\omega, \mathbf{k})_{(B)} = \frac{\sigma_0 \mathbf{k}^2}{D \mathbf{k}^2 - i\omega} - \frac{\sigma_0^2 \mathbf{k}^4}{(D \mathbf{k}^2 - i\omega)^2} \left(\frac{a_3}{a_2} + \frac{\sigma_1}{\sigma_0} \right)^2 \frac{i\omega}{32\pi D} \sqrt{\mathbf{k}^2 - \frac{2i\omega}{D}} + \mathcal{O}(\Lambda^3)$$

- **Weak** hydrostatic compatibility:

$$G^{\text{ret}}(\omega = 0, \mathbf{k}) \text{ is analytic}^* \text{ in } \mathbf{k}$$

[AJ, Kovtun, Ritz, Shukla 2020]

Long-Time Tails

Hydrodynamic modes
decay polynomially
rather than exponentially.

[Alder, Wainwright 1970]

[De Schepper, Van Beyer, Ernst 1974]

[Forster, Nelson, Stephen 1977]

[Arnold, Yaffe 1997]

[Kovtun, Yaffe 2003]

[Chen-Lin, Delacretaz, Hartnoll 2018]

*In the absence of spontaneous symmetry breaking.

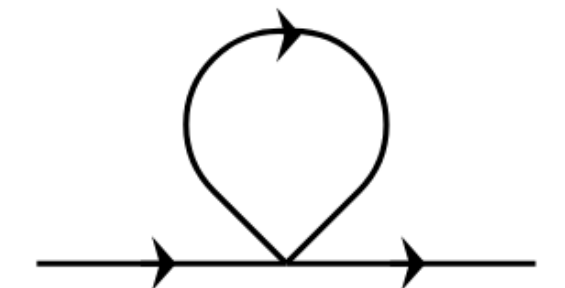
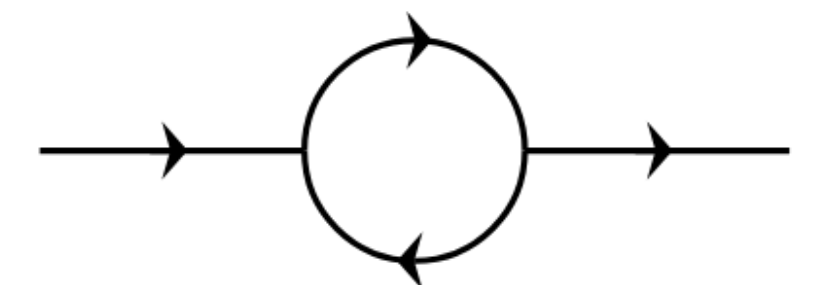
Hydrostatic compatibility

- The *correct* hydrostatic compatible results:

$$G^{\text{ret}}(\omega, \mathbf{k})_{(\text{A})} = \frac{\Gamma_0}{\Omega - i\omega} - \frac{\Gamma_0^2}{(\Omega - i\omega)^2} \left[\frac{a_4}{a_2} - \frac{2\Omega}{2\Omega - i\omega} \left(\frac{a_3}{a_2} + \frac{i\omega}{\Omega} \frac{\Gamma_1}{\Gamma_0} \right)^2 + \frac{i\omega}{\Omega} \left(\frac{\Gamma_2}{\Gamma_0} - \frac{2\Gamma_1^2}{\Gamma_0^2} \right) \right] \frac{\Lambda^3}{12\pi^2} + \dots$$

$$G^{\text{ret}}(\omega, \mathbf{k})_{(\text{B})} = \frac{\sigma_0 \mathbf{k}^2}{D \mathbf{k}^2 - i\omega} - \frac{\sigma_0^2 \mathbf{k}^4}{(D \mathbf{k}^2 - i\omega)^2} \left[\left(\frac{a_4}{a_2} - \frac{a_3^2}{a_2^2} + \frac{\sigma_2}{\sigma_0} \frac{i\omega}{D \mathbf{k}^2} \right) \frac{\Lambda^3}{12\pi^2} + \frac{i\omega}{D} \left(\frac{a_3}{a_2} + \frac{\sigma_1}{\sigma_0} \right)^2 \left(\frac{1}{32\pi} \sqrt{\mathbf{k}^2 - \frac{2i\omega}{D}} - \frac{\Lambda}{8\pi^2} \right) \right] + \dots$$

$$\xrightarrow{\omega \rightarrow 0} G^{\text{eqb}}(\mathbf{k}) = \frac{1}{a_2} - \frac{1}{a_2^2} \left(\frac{a_4}{a_2} - \frac{a_3^2}{a_2^2} \right) \frac{\Lambda^3}{12\pi^2} + \dots$$



- The SK-EFT results (GGL prescription) differ by:

$$G^{\text{ret}}(\omega, \mathbf{k})_{(\text{A})} \sim \begin{cases} (\text{Can}): 0 \\ (\text{GCan}): \frac{-2\Gamma_0^2}{(\Omega - i\omega)^2} \left(\frac{\Gamma_2}{\Gamma_0} - \frac{\Gamma_1^2}{\Gamma_0^2} \right) \frac{\Lambda^3}{12\pi^2} \end{cases} \quad G^{\text{ret}}(\omega, \mathbf{k})_{(\text{B})} \sim \begin{cases} (\text{Can}): \text{ n/a} \\ (\text{GCan}): \frac{-\sigma_0^2 \mathbf{k}^4}{(D \mathbf{k}^2 - i\omega)^2} \frac{a_3}{a_2} \frac{\sigma_1}{\sigma_0} \frac{\Lambda^3}{12\pi^2} + \dots \end{cases}$$



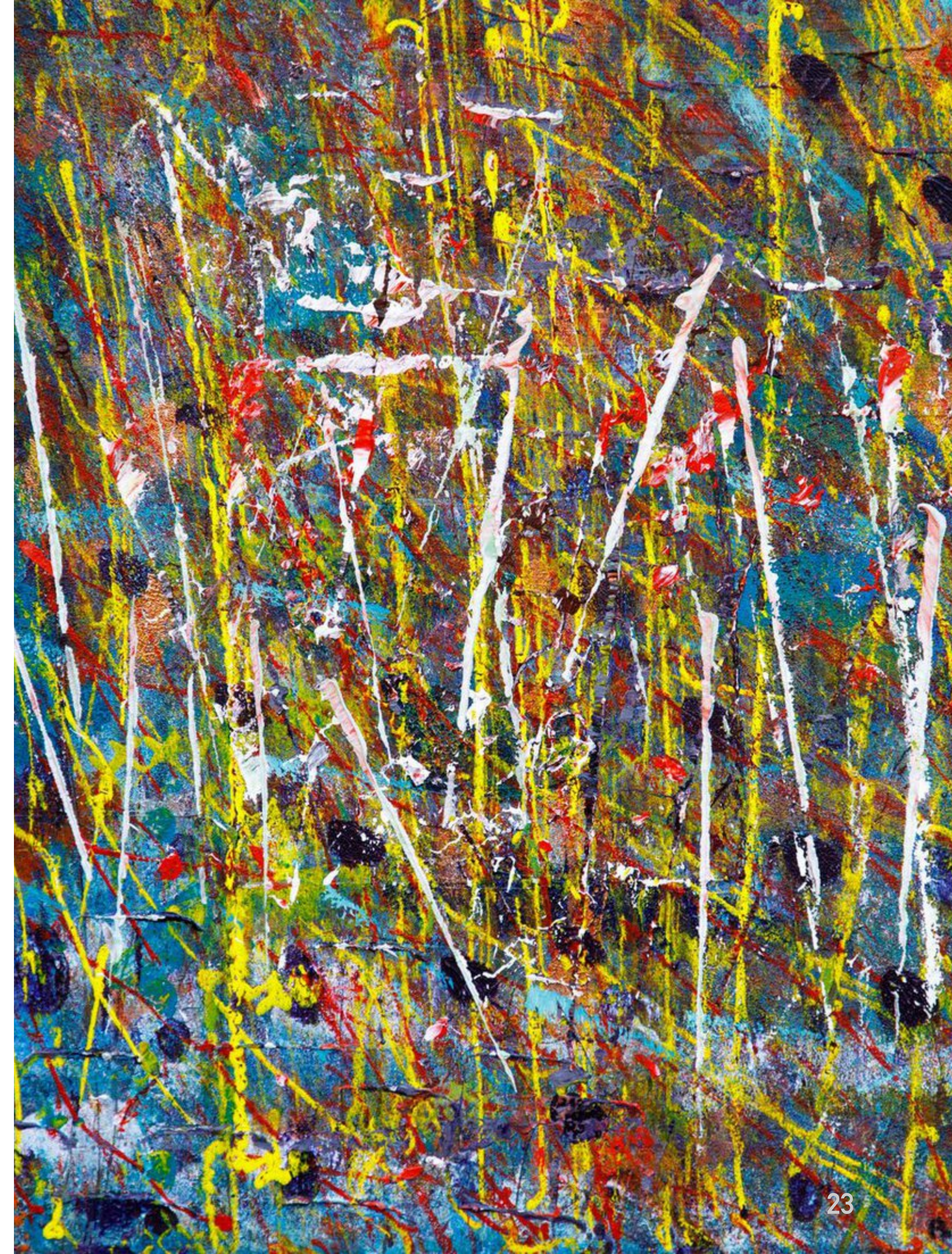
In case I made you sleep...

- Stochastic fluctuations induce loop corrections in retarded functions, which can be systmatically computed using [Schwinger-Keldysh hydrodynamics](#).
- The naive SK-EFT results are incorrect because of non-unitary diagrams. This can be fixed using the [GGL regularisation](#) prescription, at the expense of positivity of symmetric functions.
- There may be [multiple SK-EFT](#) models corresponding to the same classical hydrodynamic model.
- Hydrostatic-compatibility with GGL regularisation:

	Can Scheme (GGL)	GCan Scheme (GGL)
Model A	✓	✗
Model B	—	✗

Outline

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BRST ghosts

- The GGL prescription gives correct hydrostatic-compatible results in the “Can” scheme, but not in the “GCan” scheme.
- However, the “Can” scheme does not apply to Model B. Therefore, the GGL prescription does not yield *any* hydrostatic-compatible SK-EFT for Model B (i.e. for conserved degrees of freedom).
- The proper way to address these issues is introducing **BRST ghosts**, which systematically cancel the unphysical contributions arising from non-unitary diagrams.

[Arenas, Barci 2010]

[Haehl, Loganayagam, Rangamani 2016]

[Gao, Liu 2017]

[Jensen, Pinzani-Fokeeva, Yarom 2017]

Ghosts in MSR path integral

- Technically, the MSR generating functional involves a **functional determinant**, which gives rise to ghosts:

$$\mathcal{Z}[h_r, h_a] = \int \mathcal{D}\theta \mathcal{D}\psi_r \boxed{\det \left[\frac{\delta(E + f\theta)}{\delta\psi_r} \right]} \delta[E + f\theta] \exp \left(i \int_x \frac{i}{4} \theta^2 + h_a \psi_r \right)$$

$$\int \mathcal{D}c_r \mathcal{D}c_a \exp \left(i \int_x c_a \frac{\delta(E + f\theta)}{\delta\psi_r} c_r \right)$$

- The theory is invariant under **BRST symmetry**:

$$Q\psi_r = \epsilon c_r \quad Qc_a = \epsilon \psi_a \quad \Rightarrow \quad \boxed{\mathcal{Z}|_{h_a \rightarrow 0} = 0}$$

- The MSR ghost sector *does not* necessarily furnish a KMS symmetry.
- The ghost sector is ambiguous*: it changes if we use ϕ_r (with $\mu(\psi_r) = \partial_t \phi_r + h_r$) as a degree of freedom.

*This ambiguity is related to the well-known ambiguity in discretising stochastic Langevin equations.

Supersymmetric BRST ghosts

- Introduce **BRST ghost fields** to cancel the unphysical non-unitary contributions

Physical/Noise Fields: $\Psi_{r,a} = (\psi_{r,a}, \phi_{r,a})$

Anticommuting Ghost Fields: $C_{r,a} = (c_{r,a}, d_{r,a})$

- BRST symmetry:* $Q \Psi_r = \epsilon C_r \quad Q C_a = \epsilon \Psi_a \quad \Rightarrow \quad \mathcal{Z}|_{h_a \rightarrow 0} = 0$

- KMS symmetry: $K \Psi_r(t) = \pm \Psi_r(-t) \quad K \Psi_a(t) = \pm (\Psi_a + i\partial_t \Psi_r)(-t)$
 $K C_r(t) = \mp C_a(-t) \quad K C_a(t) = \pm C_r(-t)$

- Emergent supersymmetry: $\{Q, \bar{Q}\} = i\partial_t \quad \bar{Q} \equiv K Q K^{-1}$

Supersymmetry enables us to construct the BRST ghost sector from scratch.

[Arenas, Barci 2010]

[Haehl, Loganayagam, Rangamani 2016]

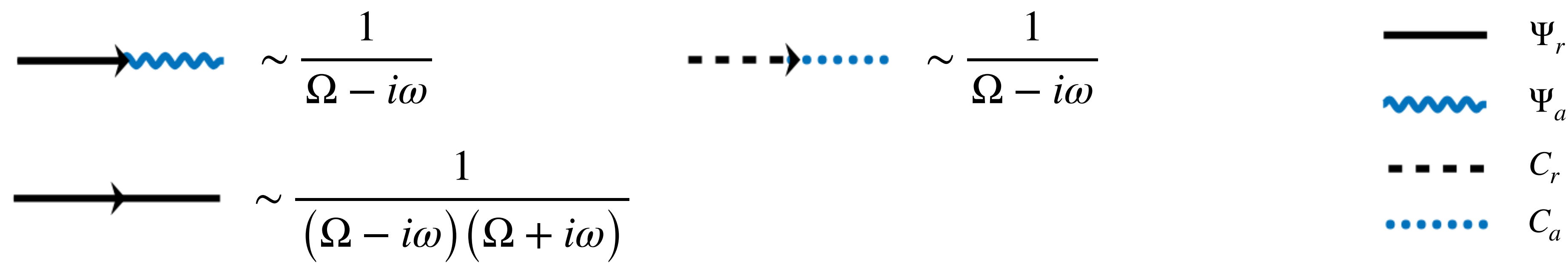
[Gao, Liu 2017]

[Jensen, Pinzani-Fokeeva, Yarom 2017]

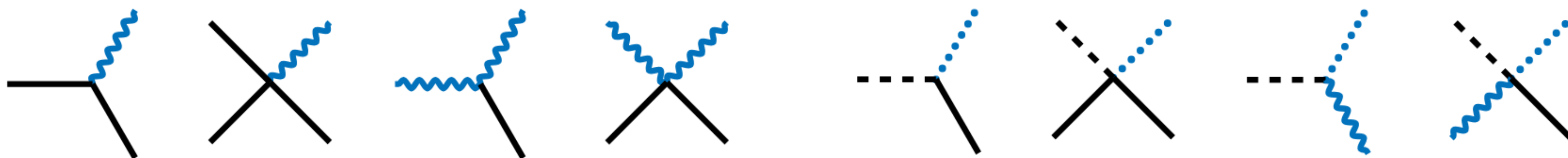
We also need to introduce “background ghost fields” that mix with physical/noise background fields under BRST transformations.

Cancellation of non-unitary diagrams

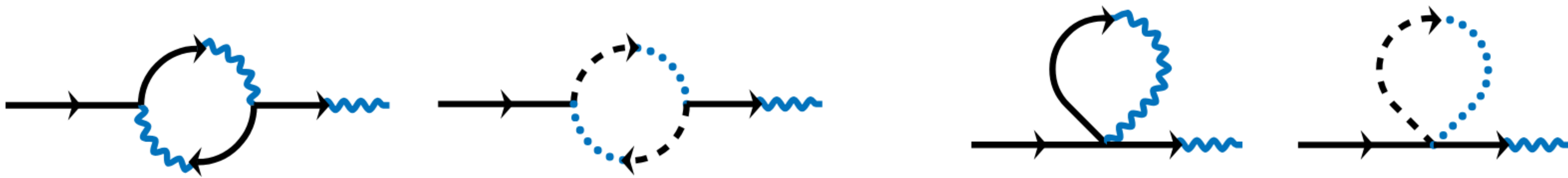
► Propagators:



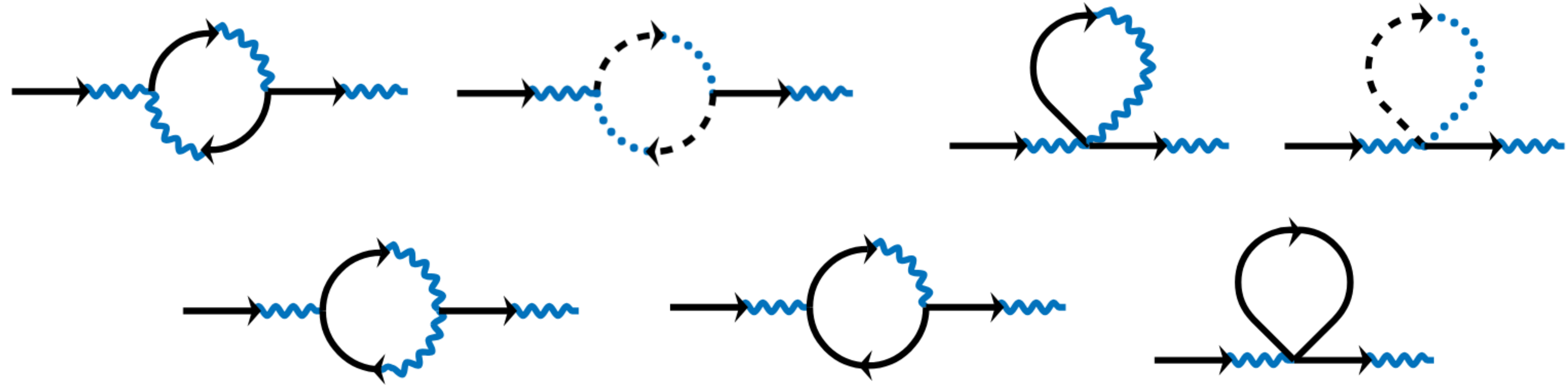
► Interactions:



► Non-unitary counter-diagrams:



Hydrostatic compatibility



► The SK-EFT results (ghost prescription) deviate from the hydrostatic-compatible results by:

$$G^{\text{ret}}(\omega, \mathbf{k})_{(A)} \sim \begin{cases} (\text{Can}): \frac{\Gamma_0^2}{(\Omega - i\omega)^2} \left(\frac{\Gamma_2}{\Gamma_0} - \frac{\Gamma_1^2}{\Gamma_0^2} \right) \frac{\Lambda^3}{12\pi^2} \\ (\text{GCan}): 0 \end{cases} \quad G^{\text{ret}}(\omega, \mathbf{k})_{(B)} \sim \begin{cases} (\text{Can}): \text{ n/a} \\ (\text{GCan}): 0 \end{cases}$$



In case I made you sleep, again...

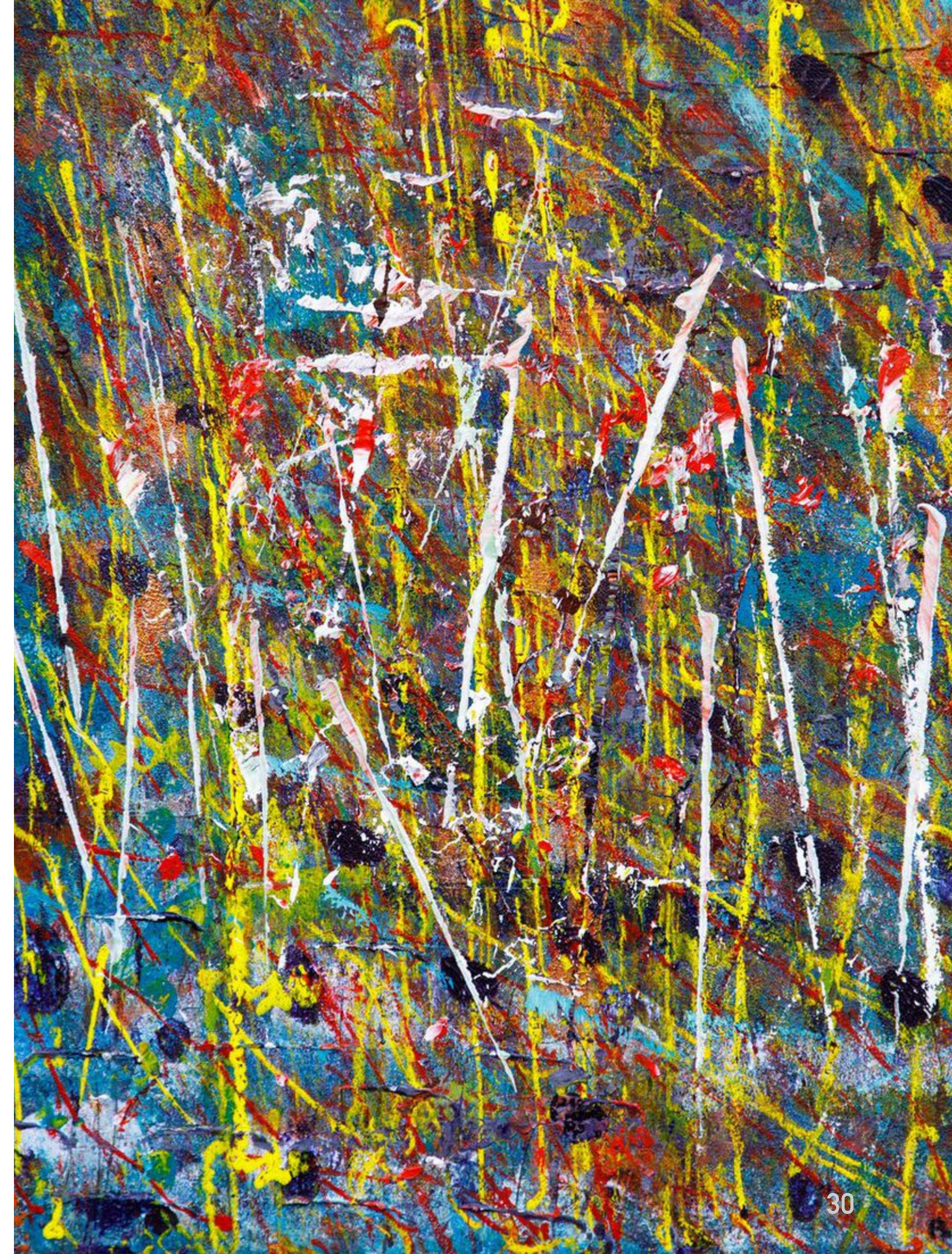
- The proper mechanism to cancel the non-unitary diagrams is by introducing **BRST ghost fields**.
- BRST symmetry combined with KMS symmetry leads to an **emergent supersymmetry** is SK-EFTs.
- Hydrostatic-compatibility with GGL and ghost regularisations:

	Can Scheme (GGL)	GCan Scheme (GGL)	Can Scheme (ghosts)	GCan Scheme (ghosts)
Model A	✓	✗	✗	✓
Model B	—	✗	—	✓

- Unfortunately, neither of the regularisation prescriptions guarantee hydrostatic-compatibility.

Outline

- Hydrostatic compatibility
- Schwinger-Keldysh hydrodynamics
- Hydrostatic compatibility at one-loop
- BRST ghosts
- **Lessons and outlook**





Lessons

- Hydrostatic compatibility of Schwinger-Keldysh hydrodynamics is not automatic. It depends on the regularisation prescription implemented to eliminate the unphysical non-unitary diagrams.
- Circumventing the need for ghosts using the GGL prescription might cause violations of unitarity in symmetric correlation functions.
- There may be multiple Schwinger-Keldysh models corresponding to the same classical hydrodynamic model.
- **What fares best:** Grand Canonical (GCan) scheme with BRST ghost regularisation is hydrostatic compatible and works for both Model A and Model B.

Hopefully, this continues to hold for full hydrodynamics with conserved energy and momentum degrees of freedom.

Wishful thinking!

Food for thought

- The “GCan” scheme breaks down near **critical points**

$$\mu(\psi) = \frac{\delta}{\delta\psi} \int_{\mathbf{x}} \left(\frac{1}{2} \partial^i \psi \partial_i \psi + \frac{a_2}{2} \psi^2 + \frac{a_4}{24} \psi^4 \right)$$

$$\begin{aligned} \mu(\psi) &= -\partial_i \partial^i \psi + \dots \\ \psi(\mu) &= -\frac{1}{\partial_i \partial^i} \mu + \dots \end{aligned}$$

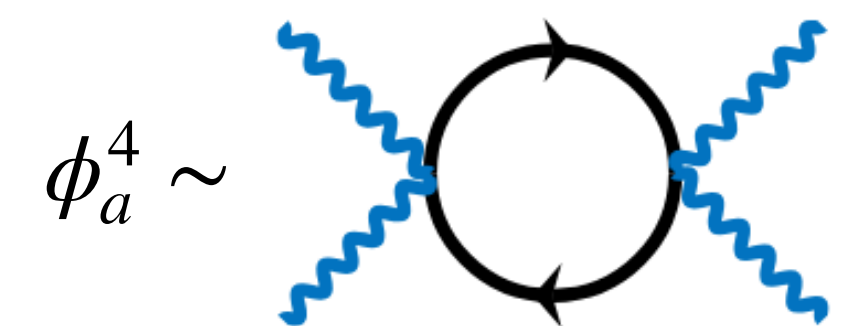
while “Can” scheme only works for Model A.

Is there a consistent hydrostatic-compatible SK-EFT for Model B?

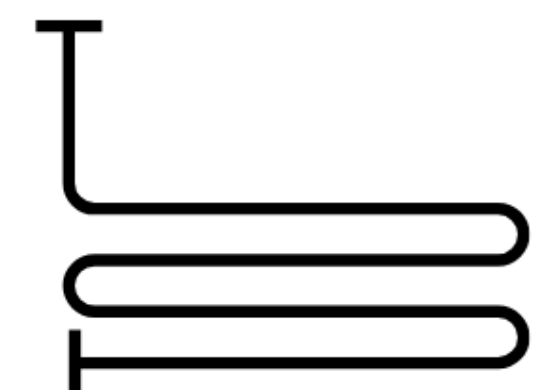
How does this fit with conventional stochastic hydrodynamics?

- Can we run real-time Wilsonian RG on SK-EFTs?
Will need **non-Gaussian stochastic transport**.

[Jain, Kovtun 2020]



- Can we extend the BRST ghost structure to **out-of-time-ordered (OTO)** SK-EFTs?
Higher- $\mathcal{N}$ emergent supersymmetry?
Interesting repucursions for **Quantum Chaos**?



THANK YOU

References

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