

Schwinger-Keldysh Hydrodynamics of the SYK Lattice



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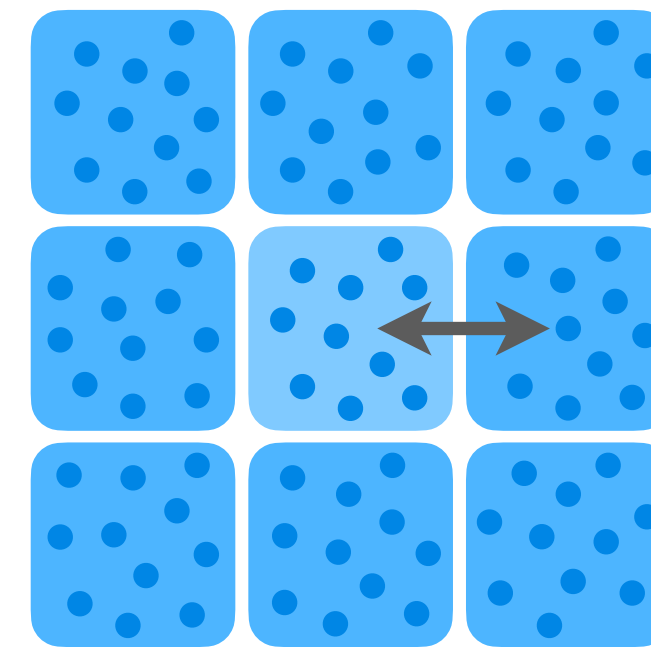
© Phil Smith "Stochastic 2"

[2604.18675] Marta Bucca, AJ, Mark Mezei, Alexey Milekhin

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EFT and Hydrodynamics

- **Effective field theory (EFT)** is a powerful tool to describe collective macroscopic phenomena in complex many-body systems, largely *agnostic* of microscopic details.
- Conventional EFTs only apply at zero temperature or in **thermal equilibrium**.
- **Hydrodynamics** describes collective dynamics of many-body systems operating “*close*” to thermal equilibrium — *local* thermal equilibrium.



- Hydrodynamics is “*almost*” an EFT — *local fields* (velocity, temperature, density, etc.), whose dynamics is dictated by universal symmetry principles.
- However, conventional hydrodynamics is *not* based on an **effective action principle**.

Schwinger-Keldysh EFT

Hydrodynamics Across Scales

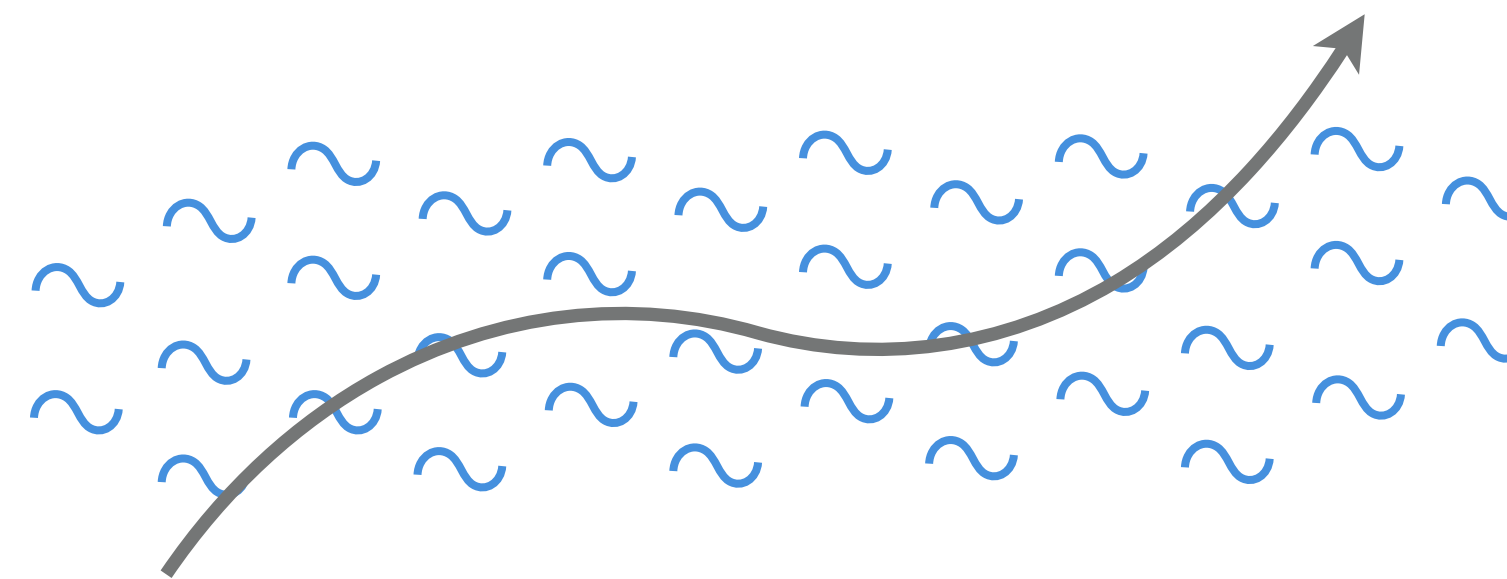
- Hydrodynamics is very **universal** — it applies to almost* all interacting physical systems in a highly excited near-equilibrium state.
- **Working assumption:** all “high-energy” excitations die out at macroscopic scales. Only collective “low-energy” excitations protected by symmetries remain:

Conserved Quantities

e.g. energy, momenta, charge, particle number

Goldstones of Spontaneously Broken Symmetries

e.g. superfluid phase, phonons



- Hydrodynamics prescribes the dynamics of low-energy excitations through universal **conservation laws** derived from symmetries.

* The fine print!

Classical Hydrodynamics

- **Classical hydrodynamics** is *deterministic*: dynamics is derived from **conservation laws**.

$$\text{Energy Conservation : } \dot{\epsilon} + \nabla \cdot \mathbf{J} = 0$$

*easily generalisable
to conserved particles
or momenta.*

- Diffusive **constitutive relations**:

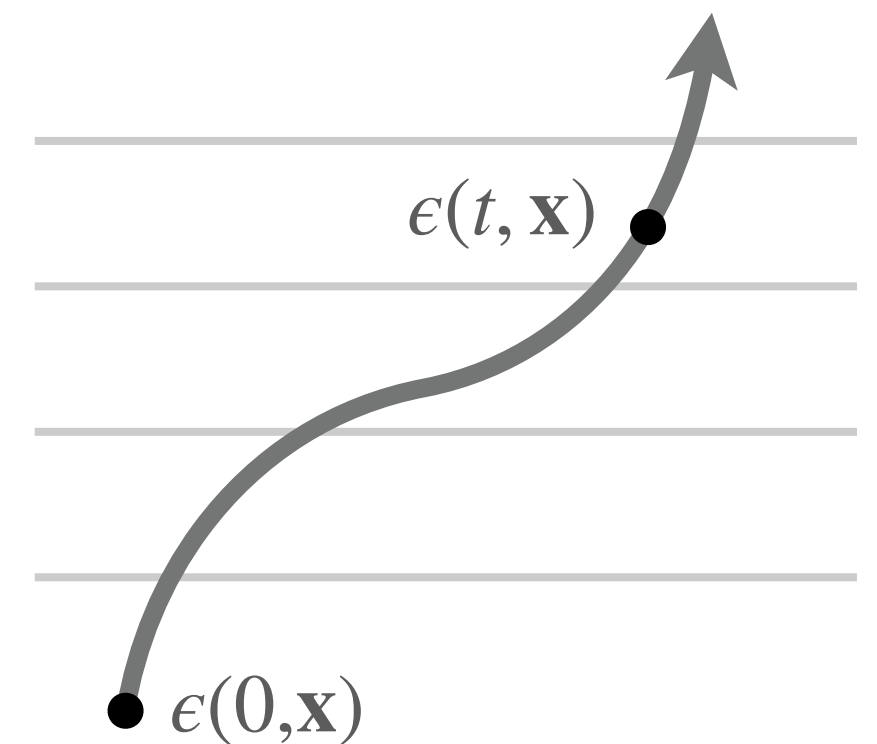
$$\mathbf{J} = -\kappa(T) \nabla T + \text{higher-derivatives}$$

thermal
conductivity

local temperature
 $T(\epsilon)$ given by the equation of state

- Energy diffusion:

$$\dot{\epsilon} = D \nabla^2 \epsilon + \text{non-linear} + \text{higher-derivatives}$$



$$D = \frac{\kappa}{c_V} \quad c_V = \frac{\partial \epsilon}{\partial T}$$

Schwinger-Keldysh EFTs

- Effective action for energy diffusion

$$\mathcal{L} = \underbrace{\left[\dot{\epsilon}(T) + \nabla \cdot \mathbf{J}(T, \nabla T, \dots) \right] X_a}_{\text{classical part}} + \underbrace{iO(X_a^2) + O(X_a^3) + iO(X_a^4) + \dots}_{\text{fluctuation part}}$$

$$\frac{\delta S}{\delta X_a} = 0 \quad \Longrightarrow \quad \dot{\epsilon} + \nabla \cdot \mathbf{J} = \# X_a$$

$$\frac{\delta S}{\delta T} = 0 \quad \Longrightarrow \quad \# X_a = 0$$

- Fluctuation part contributes to correlation functions, such as

$$\left\langle \epsilon(t, \mathbf{x}) \epsilon(t', \mathbf{x}') \dots \right\rangle \quad \left\langle \mathbf{J}(t, \mathbf{x}) \mathbf{J}(t', \mathbf{x}') \dots \right\rangle \quad \left\langle \epsilon(t, \mathbf{x}) \mathbf{J}(t', \mathbf{x}') \dots \right\rangle$$

Schwinger-Keldysh EFTs

- Effective action for energy diffusion

$$\mathcal{L} = \underbrace{\left[\dot{e}(T) + \nabla \cdot \mathbf{J}(T, \nabla T, \dots) \right] X_a}_{\text{classical part}} + \underbrace{iO(X_a^2) + O(X_a^3) + iO(X_a^4) + \dots}_{\text{fluctuation part}}$$

severely constrained by **symmetries**, **microscopic unitarity**, and the **initial state**

thermal state:
fluctuation-dissipation relations

Schwinger-Keldysh
Effective Field Theory

Schwinger-Keldysh EFT for Energy Diffusion

$$\mathcal{L} = \boxed{-\epsilon(T) \dot{X}_a + \kappa(T) \nabla T \cdot \nabla X_a + \dots} + \boxed{iT^2 \kappa(T) \nabla X_a \cdot \nabla X_a + \dots}$$

- SK EFTs are *not* manifestly unitary ($\text{Im } S \neq 0$) — strongly constrained by [microscopic unitarity](#)

$$S[T, X_a = 0] = 0 \quad S[T, -X_a] = -S^*[T, X_a] \quad \text{Im } S[T, X_a] \geq 0$$

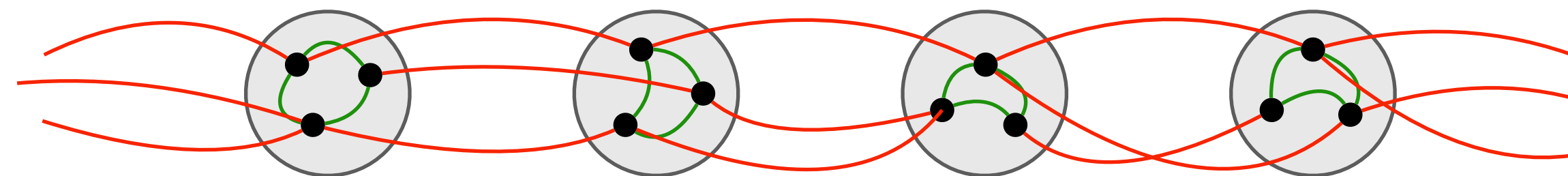
- Initial thermal states — [KMS symmetry](#) ($\beta_0 \equiv 1/T_0$)

$$T(t) \rightarrow T(-t) + O(\beta_0^2) \quad X_a(t) \rightarrow -X_a(-t) - \frac{i}{T(-t)} + i\beta_0 + O(\beta_0^2)$$

- Energy diffusion — time-translation symmetry (doubled in SK EFT)
 $\implies$ responsible for the conservation of energy (and its SK noise partner).

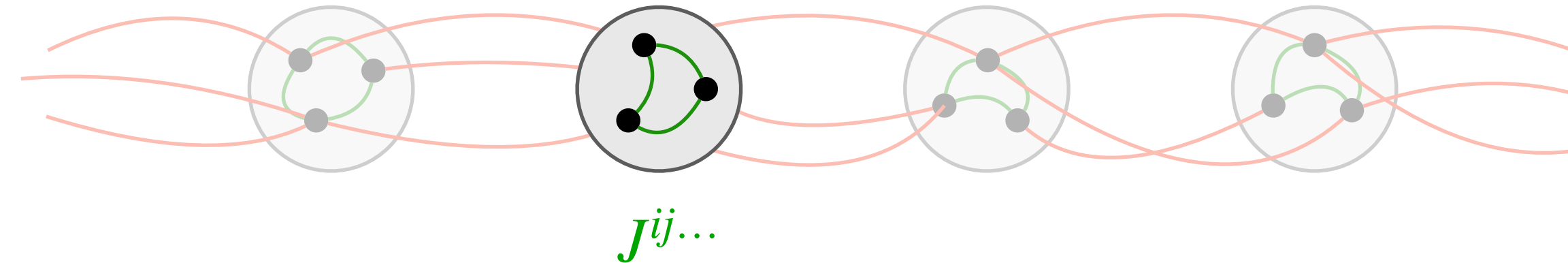
Goal for Today

To explicitly derive the SK-EFT for a simple solvable microscopic model.



SYK Chain

The SYK Dot



- **SYK dot** — 0+1d model of N Majorana fermions with local p -body random interactions.

[Sachdev, Ye 1992] [Kitaev 2015]

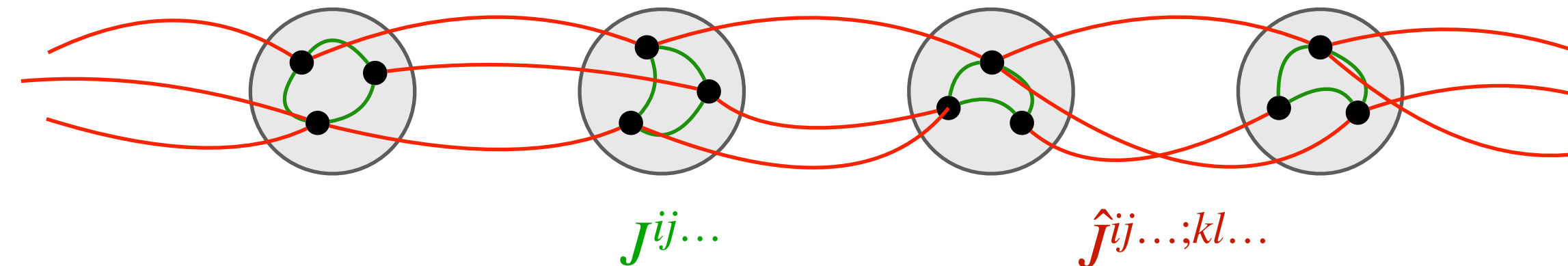
[Maldacena, Stanford 2016]

[Polchinski, Rosenhaus 2016]

Gaussian-random
couplings

$$S_{\text{SYK}} = i \sum_{i=1}^N \Psi_i \partial_t \Psi_i - i^{p/2} \sum_{i,j,\dots=1}^N J^{ij\dots} \underbrace{\left(\Psi_i \Psi_j \dots \right)}_{p \text{ fields}}$$

The SYK Chain



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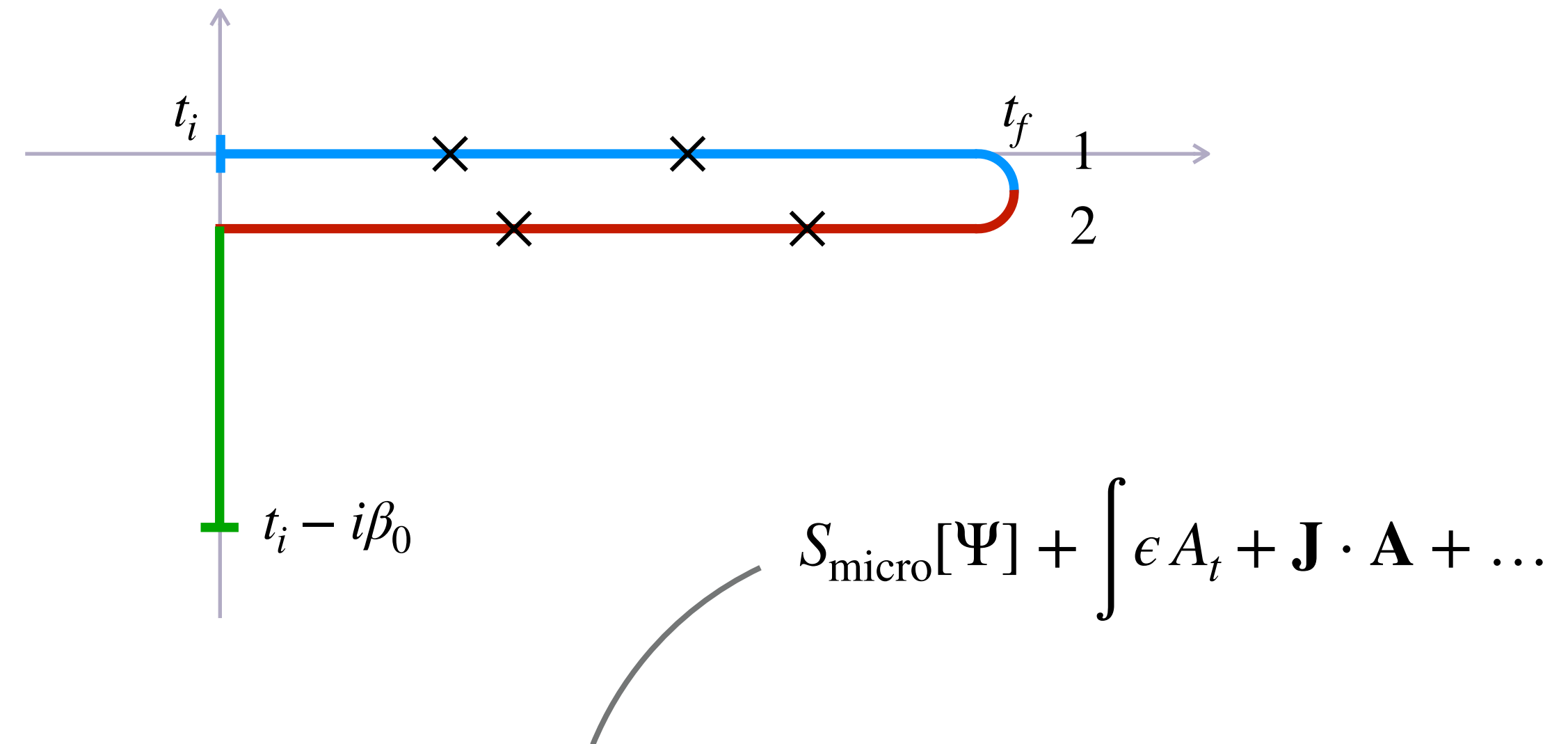
- **SYK chain** — 1+1d chain of SYK dots with random nearest-neighbour interactions.

[Gu, Qi, Stanford 2017]

$$S_{\text{SYK-chain}} = \sum_x \left[S_{\text{SYK}|x} - i^{p/2} \sum_{i,j,\dots;k,l,\dots=1}^N \hat{J}_{x,x+1}^{ij\dots;kl\dots} \underbrace{\left(\Psi_{i|x} \Psi_{j|x} \dots \right)}_{p/2 \text{ fields}} \underbrace{\left(\Psi_{k|x+1} \Psi_{l|x+1} \dots \right)}_{p/2 \text{ fields}} \right]$$

Schwinger-Keldysh Path Integral

$$\text{tr} \left(e^{-\beta H} \tilde{T}[\epsilon(t')...] T[\epsilon(t)...] \right)$$



$$\mathcal{Z}[A_1, A_2] = \int \mathcal{D}\Psi_1 \mathcal{D}\Psi_2 \mathcal{D}\Psi_E \exp \left(i S_{\text{micro}}[\Psi_1; A_1] - i S_{\text{micro}}[\Psi_2; A_2] - S_{\text{micro}}^E[\Psi_E] \right)$$

$$\Psi_1(t_f) = \Psi_2(t_f)$$

$$\Psi_2(t_i) = \Psi_E(t_i)$$

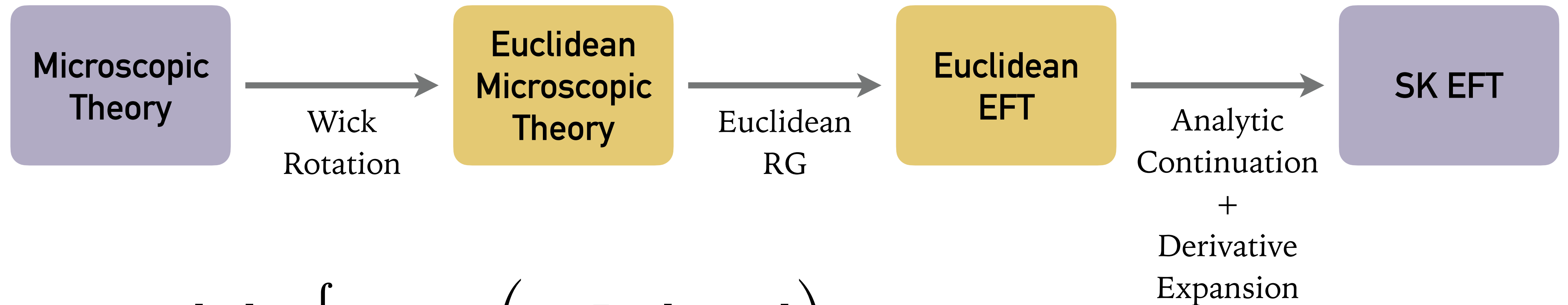
$$\Psi_1(t_i) = \Psi_E(t_i - i\beta_0)$$

RG

$$\mathcal{Z}[A_1, A_2] = \int \mathcal{D}F_1 \mathcal{D}F_2 \exp \left(i S[F_1, F_2; A_1, A_2] \right)$$

$$F_1(t_f) = F_2(t_f)$$

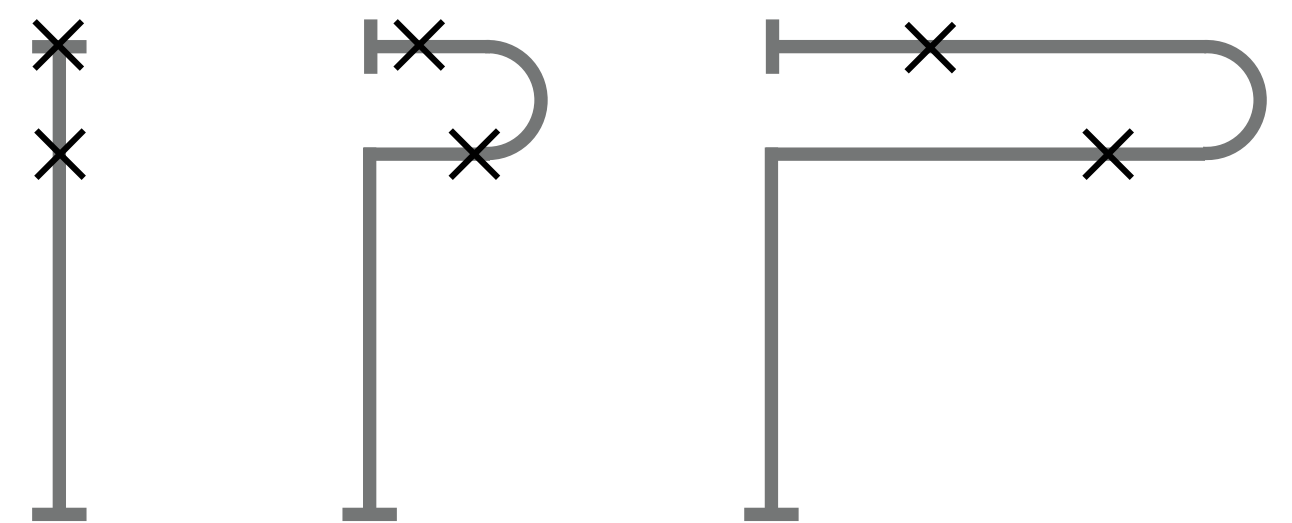
SK EFT via the Euclidean Formalism



$$\mathcal{Z}_E[A_E] = \int \mathcal{D}\Psi_E \exp\left(-S_{\text{micro}}^E[\Psi_E, A_E]\right)$$

$$\mathcal{Z}_E[A_E] = \int \mathcal{D}F_E \exp\left(i S^E[F_E; A_E]\right)$$

$$\mathcal{Z}[A_1, A_2] = \int \mathcal{D}F_1 \mathcal{D}F_2 \exp\left(i S[F_1, F_2; A_1, A_2]\right)$$



Schwarzian Theory

- Upon disorder-averaging over couplings, the SYK fermions can be traded for the **collective field**

[Maldacena, Stanford 2016]

$$G(\tau, \tau') \equiv \frac{1}{N} \sum_{i=1}^N \text{T}_E \Psi_{i,E}(\tau) \Psi_{i,E}(\tau') = \frac{1}{2} \text{sgn}(\tau - \tau') \left(1 + \frac{g(\tau, \tau') - g_0}{p} + O(1/p^2) \right)$$

- For $N \gg p^2 \gg 1$, the Euclidean path-integral is dominated by the saddle

$$\exp(g_*(\tau, \tau')) = \frac{1}{\cos^2\left(\pi v\left(1/2 - |\tau - \tau'|/\beta_0\right)\right)} \quad \beta_0 J \sim \frac{\pi v}{\cos(\pi v/2)}$$

- For $\beta_0 J \gg 1$, low-energy perturbations are governed by the **Schwarzian mode**

$$\exp(g(\tau, \tau')) \sim \frac{\dot{F}_E(\tau) \dot{F}_E(\tau')}{\sin^2\left(\pi F_E(\tau) - \pi F_E(\tau')\right)}$$

Soft Mode Action

- For sufficiently weak interactions between nearest neighbours, the low-energy description of the SYK chain is given by an **spatially-interacting Schwarzian theory**

[Maldacena, Qi 2018]

[Altland, Bagrets, Kamenev 2019]

[Almheiri, Milekhin, Swingle 2019]

[Bucca, Mezei 2024]

$$S_{\text{Sch-Chain}}^{\text{E}} = \int d\tau L_{\text{Sch}}^{\text{E}}[F(\tau)] + \int d\tau d\tau' L_{\text{int}}^{\text{E}}[F(\tau), F(\tau')]$$

$$L_{\text{Sch}}^{\text{E}}[F(\tau)] = -\mathcal{N} \sum_x \left(2\pi^2 \dot{F}(\tau, x)^2 + \frac{\ddot{F}(\tau, x)}{\dot{F}(\tau, x)} - \frac{3}{2} \frac{\dot{F}(\tau, x)^2}{\dot{F}(\tau, x)^2} \right)$$

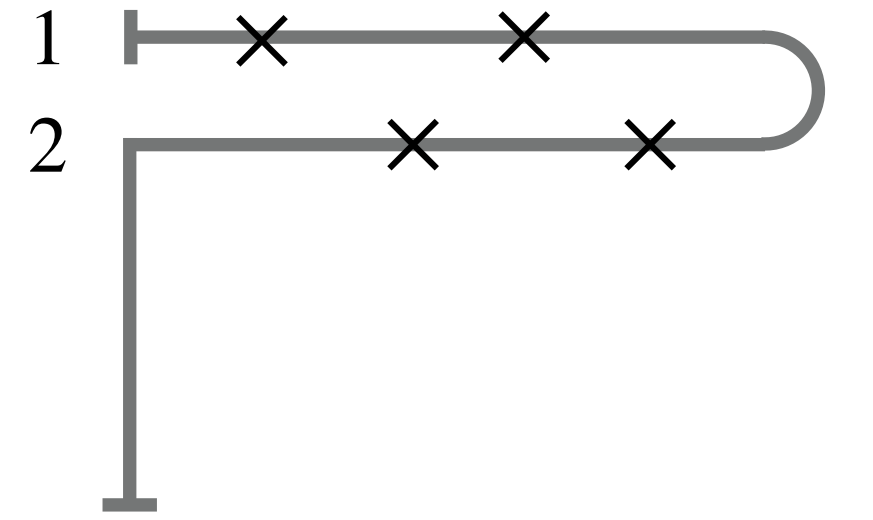
$$L_{\text{int}}^{\text{E}}[F(\tau), F(\tau')] = -\mathcal{N}' \sum_x \left(\sqrt{\frac{\dot{F}(\tau, x) \dot{F}(\tau', x)}{\sin^2(\pi F(\tau, x) - \pi F(\tau', x))}} \times (x \rightarrow x + a) \right)$$

- The soft mode action is bilocal in time.

SK-EFT for SYK Chains

- Analytic continuation to closed-time contour:

$$S = \int dt \left(L_{\text{Sch}}[F_1(t)] - L_{\text{Sch}}[F_2(t)] \right) + \int dt dt' \left(L_{\text{int}}[F_1(t), F_1(t')] - L_{\text{int}}[F_1(t), F_2(t')] - L_{\text{int}}[F_2(t), F_1(t')] + L_{\text{int}}[F_2(t), F_2(t')] \right)$$



- For $|t - t'| \gg \beta_0/\pi$, we can perform the t' integral to yield the SK EFT

$$S = \int dt \left(L_{\text{Sch}}[F_1(t)] - L_{\text{Sch}}[F_2(t)] + L_{\text{int}}[F_1(t), F_2(t)] \right)$$

Results at Leading Order in Derivatives

- At leading order* in derivatives, we recover

$$\mathcal{L} = -\epsilon(T) \dot{X}_a + \kappa(T) \nabla T \cdot \nabla X_a + iT^2 \kappa(T) \nabla X_a \cdot \nabla X_a + \dots$$

$$T = \frac{\dot{F}_1 + \dot{F}_2}{2} \quad -T X_a = F_1 - F_2$$

$$\epsilon = 2\pi^2 \mathcal{N} T^2 \quad \kappa = 8\pi^3 \mathcal{N}' T$$

Can be extended to arbitrary order in X_a and derivatives.

- Unitarity constraints and KMS symmetry:

$$S[F, F] = 0$$

$$S[F_2, F_1] = -S^*[F_1, F_2]$$

$$\text{Im } S[F_1, F_2] \geq 0$$

$$F_1(t) \rightarrow -F_1(-t)$$

$$F_2(t) \rightarrow -F_2(-t - i\beta) - i$$

(order-by-order in derivatives)

* We use the scaling $\partial_t \sim \nabla^2$ for derivatives and $\dot{F}, Q \sim O(\nabla^0)$ for fields.

Classical and Noise Energy Current

- The theory admits **two conserved energy currents**: one for each Lorentzian fold.

$$\epsilon = \frac{\epsilon_1 + \epsilon_2}{2} \quad \mathbf{J} = \frac{\mathbf{J}_1 + \mathbf{J}_2}{2}$$

classical energy current

$$\epsilon_a = \frac{\epsilon_1 - \epsilon_2}{\hbar} \quad \mathbf{J}_a = \frac{\mathbf{J}_1 - \mathbf{J}_2}{\hbar}$$

quantum/thermal noise

- The difference conservation is identically solved by $F_1 = F_2 \equiv F$.
The remaining average conservation yields the **classical energy diffusion equation**.
- The theory also admits an **entropy current** with non-negative divergence

$$\dot{s} + \nabla \cdot \mathbf{J}_s \geq 0 \quad (\text{onshell})$$

Unitarity constraints + KMS $\implies$ **local entropy production**.

Symmetries of the SK EFT

- **Average time- and space-translation symmetry** — Noether currents are purely noise:

$$F_{1,2}(t, \mathbf{x}) \rightarrow F_{1,2}(t + \xi, \mathbf{x}) \quad F_{1,2}(t, \mathbf{x}) \rightarrow F_{1,2}(t, \mathbf{x} + \mathbf{c})$$

- **Difference time-translation symmetry** — associated with average energy conservation:

$$F_1(t, \mathbf{x}) \rightarrow F_1(t + \zeta, \mathbf{x}) \quad F_2(t, \mathbf{x}) \rightarrow F_2(t - \zeta, \mathbf{x})$$

Spontaneously broken (Goldstone X_a) — **strong-to-weak SSB** by the thermal state $e^{-\beta H}$.

- **Spatially-local $SL(2, \mathbb{R})$ symmetry:** [Blake, Lee, Liu 2018] [Haehl, Rozali 2018]

$$F_{1,2}(t, \mathbf{x}) \rightarrow F_{1,2}(t, \mathbf{x}) + a(\mathbf{x}) + b_+(\mathbf{x}) e^{2\pi F_{1,2}(t, \mathbf{x})} + b_-(\mathbf{x}) e^{-2\pi F_{1,2}(t, \mathbf{x})} + O(b^2)$$

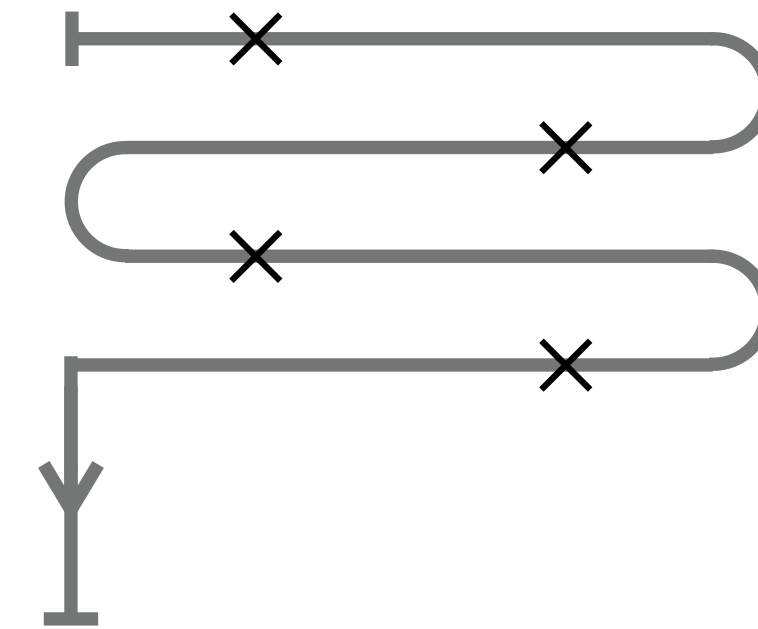
The $b_{\pm}$ parts of the symmetry are broken by the derivative expansion.

- **KMS symmetry:** $F_1(t, \mathbf{x}) \rightarrow -F_1(-t, \mathbf{x}) \quad F_2(t, \mathbf{x}) \rightarrow -F_2(-t - i\beta_0, \mathbf{x}) - i$

SK EFT for OTOCs

- SK EFT for OTOCs require **multi-fold closed-time contours**. [Mishra, Wang, Pappalardi, Delacretaz 2026]

$$\langle W(t) V(0) W(t) V(0) \rangle \sim e^{\lambda t} \quad (\text{early time})$$



$$\begin{aligned} S^{(2n)}[F_1, \dots, F_{2n}] &= \int dt \sum_{i=1}^{2n} (-1)^{i+1} L_{\text{Sch}}[F_i(t)] + \int dt dt' \sum_{i,j=1}^{2n} (-1)^{i+j} L_{\text{int}}[F_i(t), F_j(t')] \\ &= \sum_{j>i=1}^{2n} (-1)^{i+j+1} S^{(2)}[F_i, F_j] \end{aligned}$$

- **WIP:** Do OTOCs of energy admit Lyapunov growth?
- **WIP:** Large- p SYK is maximally chaotic ($\lambda = 2\pi/\beta_0$). EFT for sub-maximal chaos at finite p ?



Summary

- Derivation of dissipative SK EFT for energy diffusion from a local unitary microscopic model — SYK chain.
- Explicit identification of the low-energy variables and symmetries.
- Includes (non-Gaussian) fluctuations compatible with fluctuation-dissipation relations — realised by the (quantum) KMS symmetry.
- Access to the expectation values and correlation functions of energy and entropy currents in the SYK chain, at arbitrarily high derivative orders.
- Includes several stochastic transport coefficients not fixed by classical energy diffusion.
- Future: conserved momenta/particles, OTOCs, sub-maximal chaos, holography.



THANK YOU

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