

Stability, Causality, and Unitarity of Relativistic Schwinger-Keldysh Hydrodynamics

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The Big Picture

- **Effective field theory** is a robust framework for describing physical many-body systems without the detailed knowledge of their microscopic constituents.
- The conventional treatment of effective field theory is only suited for **zero temperature** or equilibrium phenomena at **finite temperature** (statistical field theory).
- **Hydrodynamics** is a “*universal*” low-energy effective description based on conservation laws that allows us to perturbatively depart from thermal equilibrium.
- Classical hydrodynamics has limited applicability due to the presence of **stochastic thermal fluctuations**.
- **Schwinger-Keldysh hydrodynamics** is a **symmetry-based effective field theory** generalisation of classical hydrodynamics that aims to systematically incorporate the effects of arbitrary stochastic thermal fluctuations.



Overview

- Schwinger-Keldysh Hydrodynamics¹ is a “better” way to do Hydrodynamics.
 - **Effective action / path-integral description:** cleaner to implement symmetries.
 - **Systematic inclusion of stochastic noise:** allows access to far-from-equilibrium physics, e.g. critical points, active matter.
 - **Built in: second law of thermodynamics, Onsager’s relations, (non-linear) fluctuation-dissipation theorems (FDTs).**
 - **Extends the applicability** of hydrodynamics: non-universal stochastic interactions and quantum interactions.
 - **Connections with quantum chaos.**
- Today’s discussion: how to circumvent the **stability and causality issues** in relativistic Schwinger-Keldysh hydrodynamics.

¹Crossley, Glorioso, Liu [1511.03646]
Haehl, Loganayagam, Rangamani [1511.07809]
Jensen, Pinzani-Fokeeva, Yarom [1701.07436]



Lessons

- Hydrodynamics is **not** completely characterised by its constitutive relations. New **stochastic transport coefficients**¹ “contaminate” hydrodynamic observables at higher derivative orders.
- Ensuring the 2-point FDT ($G^S \propto \text{Im } G^R$) in Martin-Siggia-Rose (MSR) formalism (stochastic hydrodynamics) **does not** guarantee consistency with higher-point FDTs.²
- Not all “stable and causal formalisms” of relativistic hydrodynamics are created equal: *Müller-Israel-Stewart (MIS) hydrodynamics* is consistent with stochastic fluctuations, while the newer *Bemfica-Disconzi-Noronha-Kovtun (BDNK) hydrodynamics*⁴ **is not**.^{2,3}
- Consistent UV-completions of hydrodynamics, like MIS hydrodynamics, may simultaneously realise **more than one** Kubo-Martin-Schwinger (KMS) symmetries.²

¹AJ, Kovtun [2009.01356]

²AJ, Kovtun [2309.00511]

³Mullins, Hippert, Gavassino, Noronha [2309.00512]

⁴Kovtun [1907.08191], Bemfica, Disconzi, Noronha [1907.12695]

Classical Hydrodynamics

- Conservation equations associated with the global symmetries

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu J^\mu = 0$$

- Hydrodynamic fields to solve for using the conservation equations

$$u^\mu \qquad T \qquad \mu$$

- Constitutive relations for the conserved currents

$$T^{\mu\nu} = \epsilon(T, \mu) u^\mu u^\nu + p(T, \mu) \Delta^{\mu\nu} - \eta(T, \mu) \Delta^{\mu\rho} \Delta^{\nu\sigma} \left(\partial_\rho u_\sigma + \partial_\sigma u_\rho - \frac{2}{d} \Delta_{\rho\sigma} \partial_\lambda u^\lambda \right) - \zeta(T, \mu) \Delta^{\mu\nu} \partial_\lambda u^\lambda + \dots$$

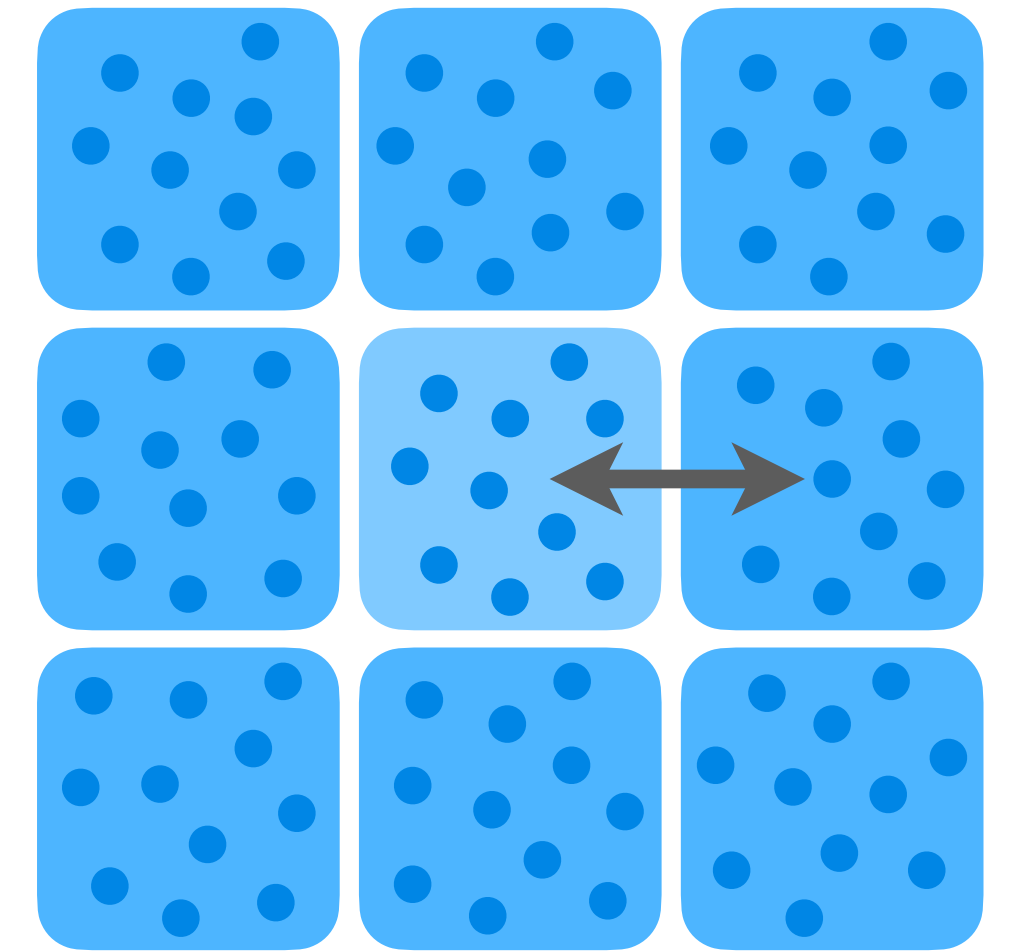
$$J^\mu = n(T, \mu) u^\mu - \sigma(T, \mu) T \Delta^{\mu\nu} \partial_\nu \frac{\mu}{T} + \dots$$

- Second law constraints:

$$\delta p = \frac{\epsilon + p}{T} \delta T + n T \delta \frac{\mu}{T} \qquad \eta, \zeta, \sigma \geq 0$$

$$u^\mu u_\mu = -1$$

$$\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$$



Mode Spectrum

- Linearised perturbations around an equilibrium state

$$\mathbf{u} = \mathbf{0} + \delta\mathbf{u}(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \quad T = T_0 + \delta T(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \quad \mu = \mu_0 + \delta\mu(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}}$$

- For simplicity, assume **probe limit** $\partial p / \partial n|_{\epsilon} = 0 \implies$ particles *do not backreact* on energy-momentum.

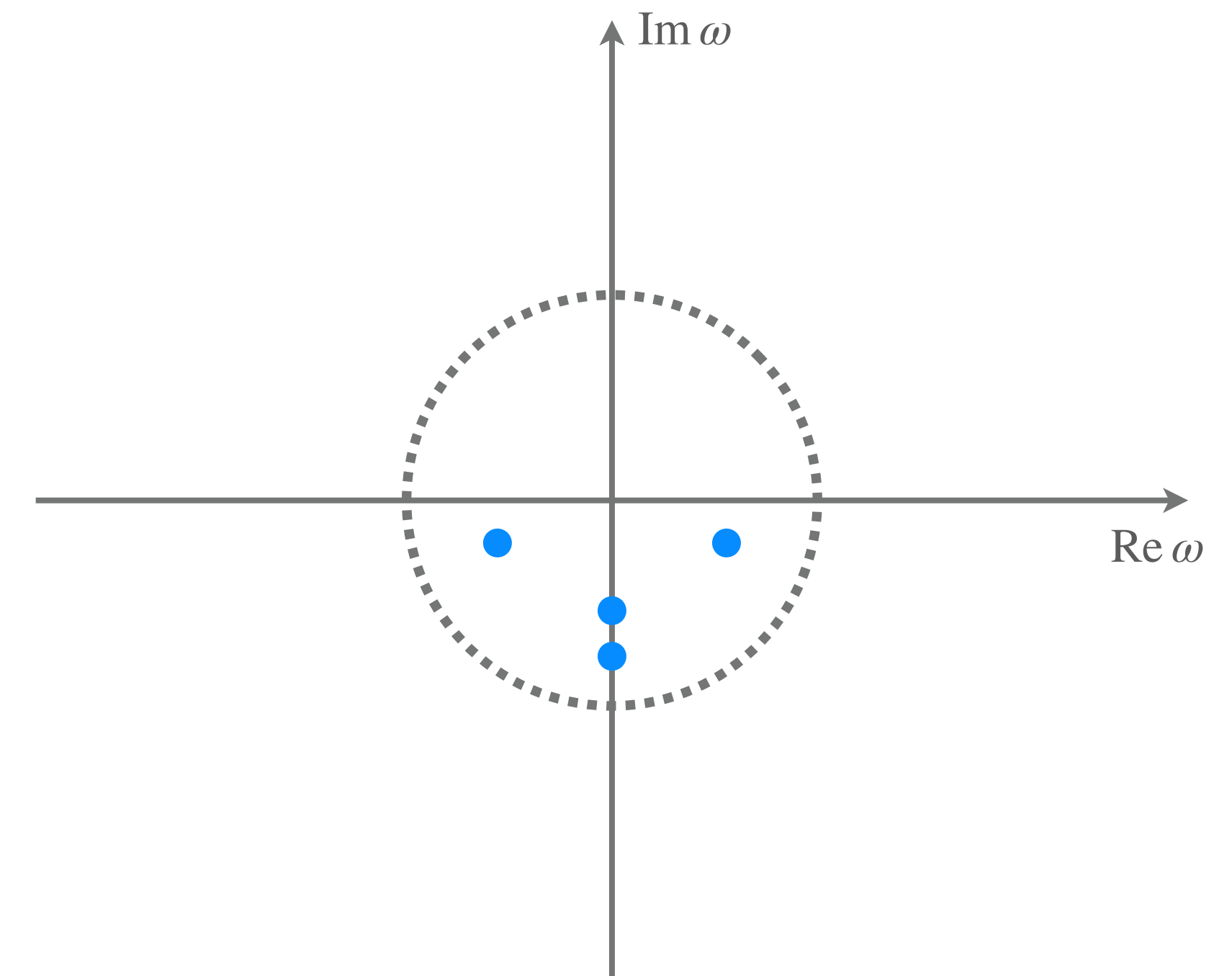
$$c_v p = \epsilon - m \times n$$

- Dispersion relations contain sound mode, shear mode, and diffusion mode

$$\delta u_{\parallel}, \delta T : \quad \omega^2 - \mathbf{k}^2 \boxed{\frac{\partial p}{\partial \epsilon} \Big|_{s/n}}_{v_s^2} + i\omega \mathbf{k}^2 \boxed{\frac{\zeta + 2\frac{d-1}{d}\eta}{\epsilon + p}}_{D_{\pi}^{\parallel}}$$

$$\delta u_{\perp} : \quad i\omega - \mathbf{k}^2 \boxed{\frac{\eta}{\epsilon + p}}_{D_{\pi}^{\perp}}$$

$$\delta\mu : \quad i\omega - \mathbf{k}^2 \boxed{\frac{\sigma}{T\partial(\mu/T)/\partial n|_{\epsilon}}}_{D_n}$$



Boosted Modes – Instabilities and Acausalities

- Modes in a boosted state can be obtained using

$$\omega \rightarrow u_0^\mu p_\mu \quad \mathbf{k}^2 \rightarrow \Delta^{\mu\nu} p_\mu p_\nu$$

- This generates **3 new unstable modes** *outside* the hydrodynamic regime

$$\omega = i \frac{\#}{\mathbf{v}^2} \frac{1}{D_\pi^\parallel} + \dots \quad \omega = i \frac{\#}{\mathbf{v}^2} \frac{1}{D_\pi^\perp} + \dots \quad \omega = i \frac{\#}{\mathbf{v}^2} \frac{1}{D_n} + \dots$$

- The modes propagate superluminally at large wavevector

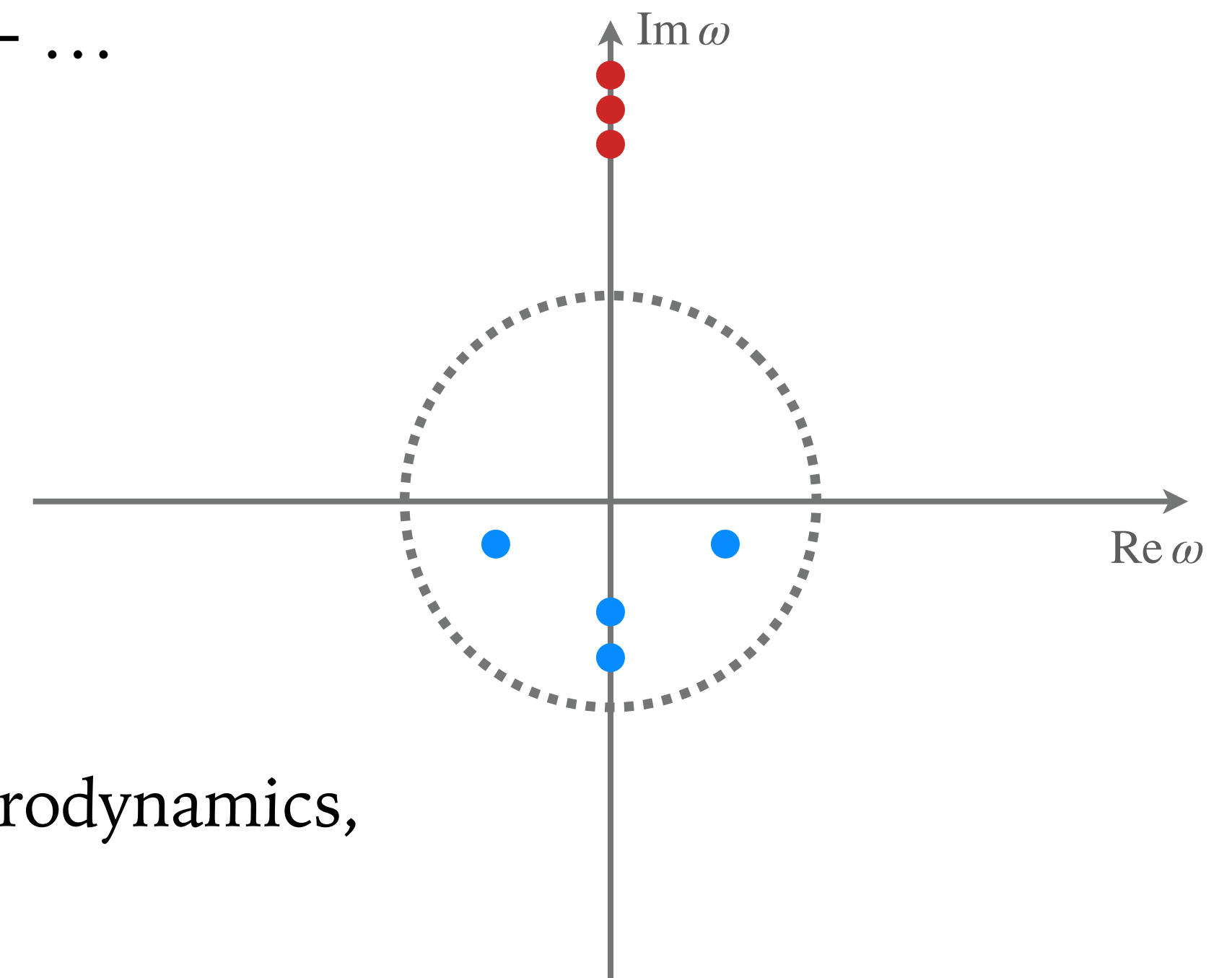
$$\lim_{k \rightarrow \infty} \left| \text{Re} \frac{\partial \omega}{\partial \mathbf{k}} \right| = \left| \frac{\mathbf{v} \cdot \hat{\mathbf{k}}}{\mathbf{v}^2} \right| \not\leq 1$$

- While these “pathologies” lie outside the applicability regime hydrodynamics, they render relativistic hydrodynamics inapt for solving for flows.

$$p^\mu = (\omega, \mathbf{k})$$

$$u^\mu = \frac{(1, \mathbf{v})}{\sqrt{1 - \mathbf{v}^2}}$$

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}$$



Adding a UV Regulator

- Hereafter, we will focus *exclusively* on the U(1) sector.
- Regulate diffusion dispersion relations with a new stable gapped mode

$$i\omega - \mathbf{k}^2 D_n \longrightarrow i\omega (1 - i\omega \tau) - \mathbf{k}^2 D_n$$

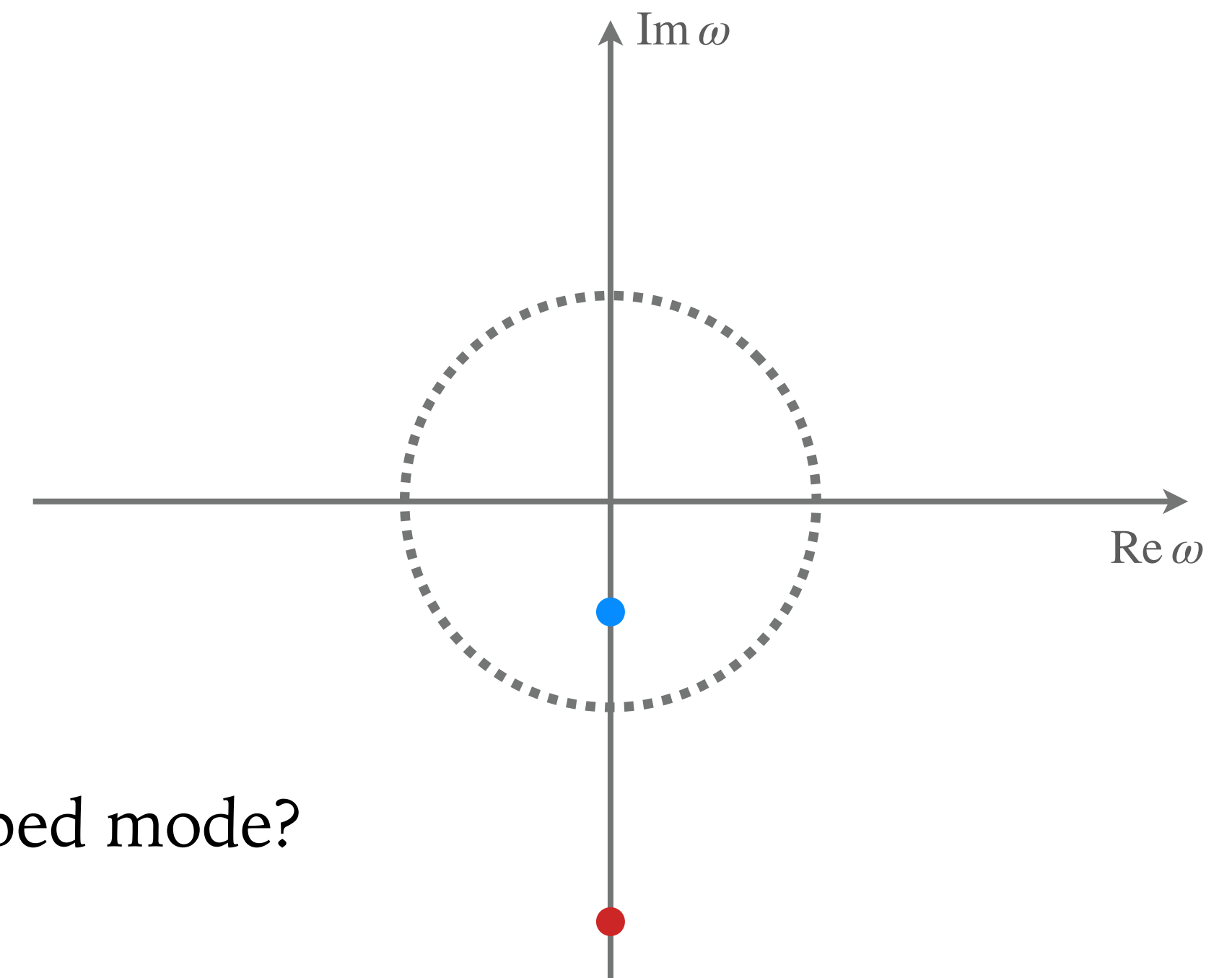
with a new pole $\omega = -i/\tau + \dots$

- The boosted spectrum remains stable as long as $\tau \geq D_n$

$$\omega = -i \frac{\#}{\tau - \mathbf{v}^2 D_n} + \dots$$

$$\lim_{k \rightarrow \infty} \left| \text{Re} \frac{\partial \omega}{\partial \mathbf{k}} \right| = \left| \frac{\tau - D_n}{\tau - \mathbf{v}^2 D_n} \mathbf{v} \cdot \hat{\mathbf{k}} \right| \leq 1$$

- How can we extend relativistic hydrodynamics to include the gapped mode?



MIS vs BDNK Hydrodynamics

$$J^\mu = n u^\mu + \mathcal{J}^\mu$$

MIS hydrodynamics

- Replace flux with a new relaxed variable

$$\mathcal{J}^\mu + \tau \Delta^{\mu\nu} u^\lambda \partial_\lambda \mathcal{J}_\nu = -\sigma \Delta^{\mu\nu} \partial_\nu \mu + \dots$$

- *Pro*: hydrodynamic differential equations are first-order in time-derivatives.
- *Con*: requires introducing new variables.

BDNK hydrodynamics

- Use a redefinition of hydrodynamic variables

$$\mu \rightarrow \mu + \tau u^\mu \partial_\mu \mu$$

$$\mathcal{J}^\mu = \tau u^\mu u^\nu \partial_\nu n - \sigma \Delta^{\mu\nu} \partial_\nu \mu + \dots$$

- *Pro*: no need to introduce a new variable.
- *Con*: hydrodynamic equations involve second-order time-derivatives.
- *Con*: inconsistent with stochastic fluctuations.

¹Muller (1967), Israel-Stewart (1976)

²Bemfica-Disconzi-Noronha [1907.12695], Kovtun [1907.08191]

Retarded Correlation Functions

- Constitutive relations with background sources

$$J^\mu[\mu, A] = n(\mu) u^\mu - \sigma(\mu) \Delta^{\mu\nu} (\partial_\nu \mu - E_\nu) + \dots$$

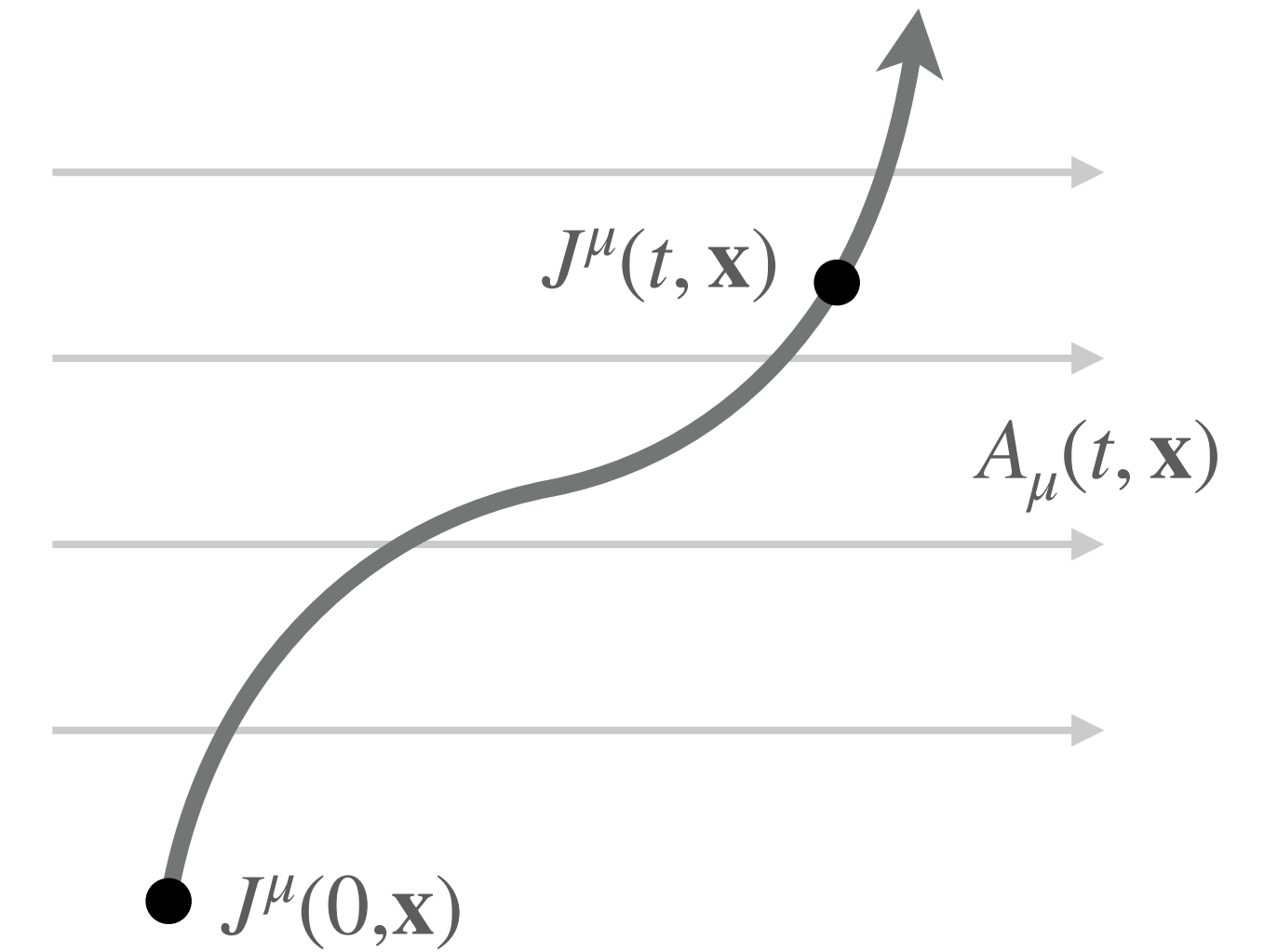
- Consider the solution of $\partial_\mu J^\mu = 0 \implies \mu_{\text{cl}}[A]$

$$J_{\text{cl}}^\mu[A] \equiv J^\mu[\mu_{\text{cl}}[A], A]$$

- Response functions or retarded correlation functions

$$J_{\text{cl}}^\mu(p)[A] = J_0 + \int_{p'_1} G_{J^\mu J^\nu}^{\text{R}}(p; p'_1) \delta A_\nu(p'_1) + \frac{1}{2} \int_{p'_1, p'_2} G_{J^\mu J^\nu J^\rho}^{\text{R}}(p; p'_1, p'_2) \delta A_\nu(p'_1) \delta A_\rho(p'_2) + \dots$$

$$G_{J^\mu J^\nu J^\rho \dots}^{\text{R}}(p; p'_1, p'_2, \dots) = \left(\frac{\delta}{\delta A_\nu(p'_1)} \frac{\delta}{\delta A_\rho(p'_2)} \dots \right) J_{\text{cl}}^\mu(p)[A]$$



$$E_\mu \equiv (\partial_\mu A_\nu - \partial_\nu A_\mu) u^\nu$$

Symmetric Correlation Functions via FDTs

- Fluctuation-dissipation theorems (FDTs) determine the symmetric correlation functions

$$\begin{aligned} G_{J^\mu J^\nu}^S(p_1, p_2) &\equiv \langle J^\mu(p_1) J^\nu(p_2) \rangle - \langle J^\mu(p_1) \rangle \langle J^\nu(p_2) \rangle && \text{“Variance of } J^\mu\text{”} \\ &= \frac{1}{i\beta_0 \omega_1} \left(G_{J^\mu J^\nu}^R(p_1; p_2) - G_{J^\mu J^\nu}^{R*}(p_1; p_2) \right) \end{aligned}$$

$$\begin{aligned} G_{J^\mu J^\nu J^\rho}^S(p_1, p_2, p_3) &\equiv \langle J^\mu(p_1) J^\nu(p_2) J^\rho(p_3) \rangle - \langle J^\mu(p_1) J^\nu(p_2) \rangle \langle J^\rho(p_3) \rangle + \langle J^\mu(p_1) \rangle \langle J^\nu(p_2) \rangle \langle J^\rho(p_3) \rangle \\ &\quad - \left(\langle J^\mu(p_1) \rangle \langle J^\nu(p_2) J^\rho(p_3) \rangle + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \right) \\ &= \frac{1}{(i\beta_0 \omega_2)(i\beta_0 \omega_3)} \left(G_{J^\mu J^\nu J^\rho}^R(p_1; p_2, p_3) + G_{J^\mu J^\nu J^\rho}^{R*}(p_1; p_2, p_3) \right) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \end{aligned}$$

- G^S probes the **probabilistic nature** of observables due to **thermal noise** and cannot be intrinsically computed using classical hydrodynamics.

¹Similar theorems exist for higher-point functions, but involve partially-retarded correlation functions [Wang, Heinz [hep-th/9809016]]

Regulated Correlation Functions

► Ordinary hydrodynamics

$$G_{J_t J_t}^R(p; -p) = \frac{-\mathbf{k}^2 \sigma}{i\omega - \mathbf{k}^2 D} + \dots$$

$$G_{J_t J_t}^S(p, -p) = \frac{2}{\beta_0} \frac{\mathbf{k}^2 \sigma}{|i\omega - \mathbf{k}^2 D|^2} + \dots$$

► MIS hydrodynamics

$$G_{J_t J_t}^R(p; -p) = \frac{-\mathbf{k}^2 \sigma}{i\omega (1 - i\omega \tau) - \mathbf{k}^2 D} + \dots$$

$$G_{J_t J_t}^S(p, -p) = \frac{2}{\beta_0} \frac{\mathbf{k}^2 \sigma}{|i\omega (1 - i\omega \tau) - \mathbf{k}^2 D|^2} + \dots$$

► BDNK hydrodynamics

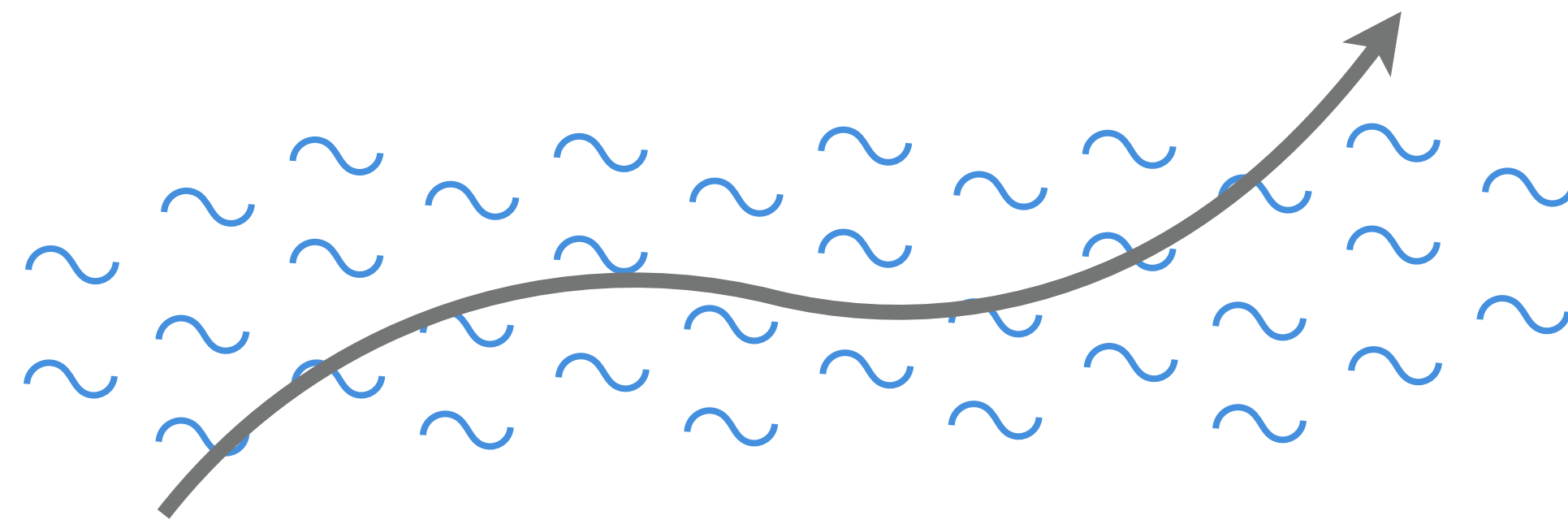
$$G_{J_t J_t}^R(p; -p) = \frac{-(1 - i\omega \tau) \mathbf{k}^2 \sigma}{i\omega(1 - i\omega \tau) - D \mathbf{k}^2} + \dots$$

$$G_{J_t J_t}^S(p, -p) = \frac{2}{\beta_0} \frac{(1 + \omega^2 \tau^2) \mathbf{k}^2 \sigma \boxed{-\mathbf{k}^4 D \sigma \tau}}{|i\omega(1 - i\omega \tau) - D \mathbf{k}^2|^2} + \dots$$

Variance is not positive-definite.

Stochastic Fluctuations

- The **classical** formulation of hydrodynamics based on conservation equations might be too simple for many purposes.
- This accounts for mutual interactions between conserved variables, but not for possible interactions with the background **thermal noise** arising due to the ignored microscopic excitations.



- These effects lead to physically observable signatures in the low-energy spectrum, known as long-time tails, that cannot be predicted by classical hydrodynamics.¹

¹[Alder, Wainwright (1970)] [De Schepper, Van Beyeren, Ernst (1974)]
[Forster, Nelson, Stephen (1977)]

Stochastic Hydrodynamics – MSR Formalism

- Hydrodynamic observables are **stochastic** due to thermal noise

$$\langle J^\mu J^\nu \dots \rangle \neq \langle J^\mu \rangle \langle J^\nu \rangle \dots$$

- Introduce arbitrary

$$J^\mu[\mu, A, \theta] = n(\mu) u^\mu + \mathcal{J}^\mu[\mu, A, \theta]$$

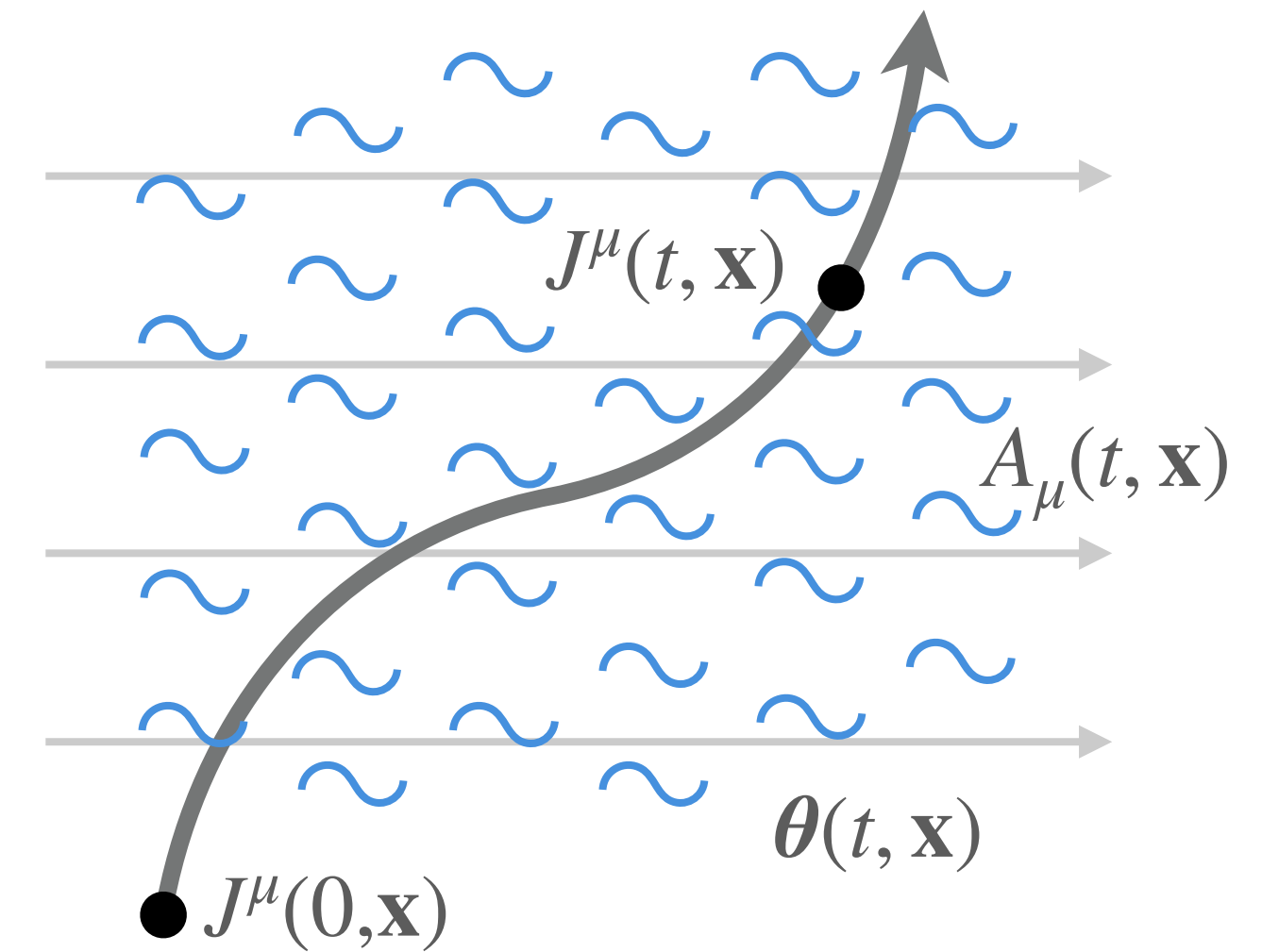
$$\text{BDNK : } \mathcal{J}^\mu = \tau u^\mu u^\nu \partial_\nu n - \sigma \Delta^{\mu\nu} (\partial_\nu \mu - E_\nu) + \theta^\mu$$

$$\text{MIS : } \mathcal{J}^\mu = -\tau \Delta^{\mu\nu} u^\lambda \partial_\lambda \mathcal{J}_\nu - \sigma \Delta^{\mu\nu} (\partial_\nu \mu - E_\nu) + \theta^\mu$$

- Integrate over random noise distribution¹ $\mathcal{P}[A, \theta]$

$$\langle J^\mu J^\nu \dots \rangle[A] = \int \mathcal{D}\theta \left(J_{\text{cl}}^\mu[A, \theta] J_{\text{cl}}^\nu[A, \theta] \dots \right) \mathcal{P}[A, \theta]$$

- $\mathcal{P}[A, \theta]$ must be consistent with FDTs.



$$J_{\text{cl}}^\mu[A, \theta] \equiv J^\mu[\mu_{\text{cl}}[A, \theta], A, \theta]$$

Generating Functional and KMS Symmetry

- We can define the non-equilibrium generating functional

$$\mathcal{Z}[A, A_a] = \int \mathcal{D}\theta \exp \left(i \int_x J_{\text{cl}}^\mu[A, \theta] A_{a\mu} \right) \mathcal{P}[A, \theta]$$

$$G_{rr\dots aa\dots}^{\mu\nu\dots\rho\sigma\dots} = \left(\frac{\delta}{\delta A_\rho} \frac{\delta}{\delta A_\sigma} \right) \langle J^\mu J^\nu \dots \rangle_{\text{conn}}[A] = \left(\frac{\delta}{\delta A_\rho} \frac{\delta}{\delta A_\sigma} \right) \left(\frac{-i\delta}{\delta A_{a\mu}} \frac{-i\delta}{\delta A_{a\nu}} \dots \right) \ln \mathcal{Z}[A, A_a]$$

- FDT constraints are succinctly summarised as the KMS symmetry¹

$$G_{JJ\dots}^{\text{R}} = G_{ra\dots}$$

$$G_{JJ\dots}^{\text{S}} = G_{rr\dots}$$

$$\mathcal{Z}[A(x), A_a(x)] = \mathcal{Z}[A(-x), (A_a + i\mathcal{L}_\beta A)(-x)]$$

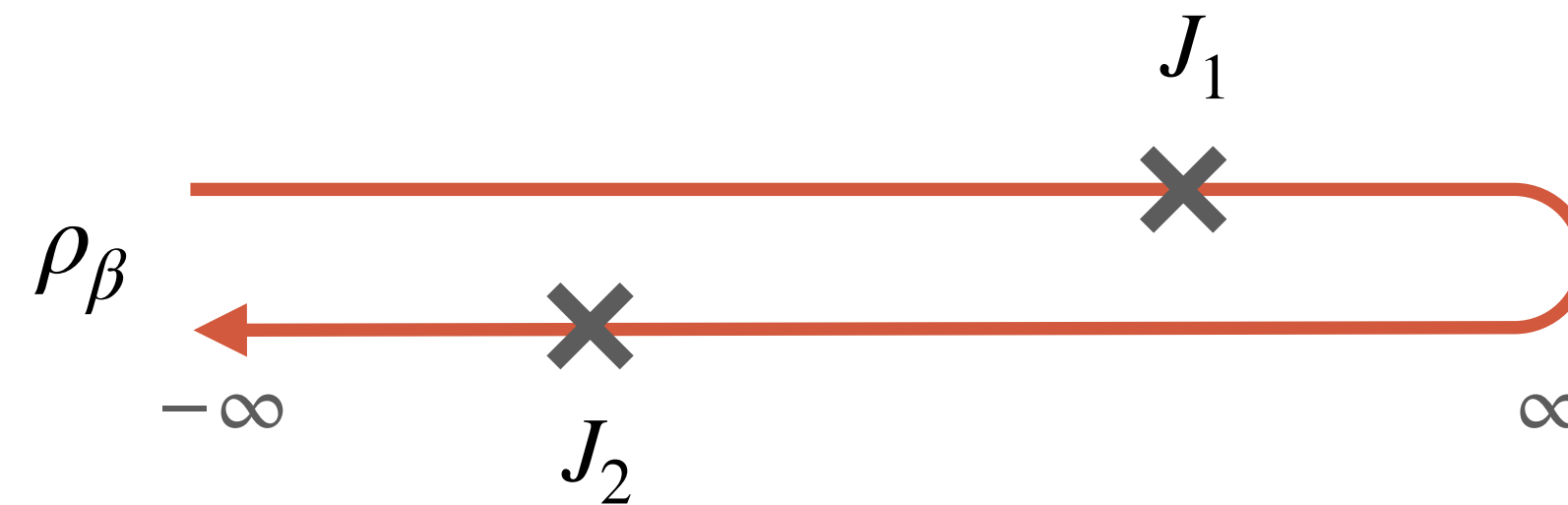
$$\mathcal{L}_\beta \equiv \beta_0 u^\mu \partial_\mu$$

- This imposes strong constraints on $\mathcal{P}[A, \theta]$.

¹This assumes theory to be PT-invariant. There are analogous versions for T, CT, CPT symmetries.

Generating Functional and KMS Symmetry

► Closed-time path integral¹



$$\mathcal{Z}[A_1, A_2] = \text{tr}_C \left[\rho \exp \left(\frac{i}{\hbar} \int_x J_1 A_1 - J_2 A_2 \right) \right]$$

$$\mathcal{Z}[A_r, A_a] = \text{tr}_C \left[\rho \exp \left(i \int_x J A_a - J_a A \right) \right]$$

$$f = \frac{f_1 + f_2}{2}$$

$$f_a = \frac{f_1 - f_2}{\hbar}$$

► KMS symmetry

$$A_1(x) \rightarrow A_1(-x)$$

$$A_2(x) \rightarrow A_2(-x - i\hbar\beta_0)$$

$\Rightarrow$

$$A(x) \rightarrow A(-x) + \mathcal{O}(\hbar)$$

$$A_a(x) \rightarrow \left(A_a + i\beta_0 \partial_t A \right)(-x) + \mathcal{O}(\hbar)$$

¹Field Theory of Non-Equilibrium Systems (Textbook), Kamenev (2011)

Non-equilibrium Effective Action

► Take Gaussian ansatz:

$$\mathcal{P}[A, \theta] = \exp \left(-\frac{1}{4} \int_x C^{\mu\nu} [\mu_{\text{cl}}[A, \theta], A] \theta_\mu \theta_\nu \right)$$

$$\mathcal{Z}[A, A_a] = \int \mathcal{D}\theta \exp \left(i \int_x J_{\text{cl}}^\mu [A, \theta] A_{a\mu} \right) \exp \left(-\frac{1}{4} \int_x C^{\mu\nu} [\mu_{\text{cl}}[A, \theta], A] \theta_\mu \theta_\nu \right)$$

$$= \int \mathcal{D}\theta \mathcal{D}\varphi_a \mathcal{D}\psi_a \mathcal{D}\mu \mathcal{D}\mathcal{J} \exp \left(i \int_x (n u^\mu + \mathcal{J}^\mu) A_{a\mu} \right) \exp \left(-\frac{1}{4} \int_x C^{\mu\nu} [\mu, A] \theta_\mu \theta_\nu \right)$$

$$\exp \left(i \int_x -\varphi_a \partial_\mu (n u^\mu + \mathcal{J}^\mu) - \psi_a^\mu \left(\mathcal{J}_\mu + \text{blob}_\mu + \theta_\mu \right) \right)$$

$$B_{a\mu} \equiv \partial_\mu \varphi_a + A_{a\mu}$$

$$= \int \mathcal{D}\varphi_a \mathcal{D}\psi_a \mathcal{D}\mu \mathcal{D}\mathcal{J} \exp \left(i \int_x \underbrace{n u^\mu B_{a\mu} + \mathcal{J}^\mu (B_{a\mu} - \psi_{a\mu}) - \psi_a^\mu \text{blob}_\mu}_{\mathcal{L}} + i C_{\mu\nu}^{-1} [\mu, A] \psi_a^\mu \psi_a^\nu \right)$$

Schwinger-Keldysh Ordinary Hydrodynamics

$$\mathcal{L} = n u^\mu B_{a\mu} + \mathcal{J}^\mu (B_{a\mu} - \psi_{a\mu}) - \sigma \Delta^{\mu\nu} \psi_{a\mu} (\partial_\nu \mu - E_\nu) + i C_{\mu\nu}^{-1} \psi_a^\mu \psi_a^\nu$$

$$B_{a\mu} \equiv \partial_\mu \varphi_a + A_{a\mu}$$

$$= n u^\mu B_{a\mu} - \sigma \Delta^{\mu\nu} B_{a\mu} (\partial_\nu \mu - E_\nu) + i C_{\mu\nu}^{-1} B_a^\mu B_a^\nu$$

► KMS symmetry:

$$\mu(x) \rightarrow \mu(-x) \quad \varphi_a(x) \rightarrow - \left(\varphi_a + i\beta_0 (\mu - u^\mu A_\mu) \right)(-x)$$

$$B_{a\mu}(x) \rightarrow \left(B_{a\mu} + i\beta_0 (\partial_\mu \mu - E_\mu) \right)(-x)$$

$$\delta\Omega = n \delta\mu$$

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu \left(i\beta_0 u^\mu \Omega \right)$$

$$C_{\mu\nu}^{-1} = T_0 \sigma \Delta_{\mu\nu}$$

► SK variables φ, φ_a

$$\varphi(x) \rightarrow -\varphi(-x) \quad \varphi_a(x) \rightarrow - \left(\varphi_a + i\mathcal{E}_\beta \varphi \right)(-x) \quad \mu \equiv u^\mu (\partial_\mu \varphi + A_\mu) \quad \varphi \rightarrow \varphi + f(\mathbf{x})$$

Schwinger-Keldysh BDNK Hydrodynamics

$$\mathcal{L} = n u^\mu B_{a\mu} + \mathcal{J}^\mu (B_{a\mu} - \psi_{a\mu}) - (\sigma \Delta^{\mu\nu} - \lambda u^\mu u^\nu) \psi_{a\mu} (\partial_\nu \mu - E_\nu) + i C_{\mu\nu}^{-1} \psi_a^\mu \psi_a^\nu$$

$$= n u^\mu B_{a\mu} - (\sigma \Delta^{\mu\nu} - \lambda u^\mu u^\nu) B_{a\mu} (\partial_\nu \mu - E_\nu) + i C_{\mu\nu}^{-1} B_a^\mu B_a^\nu$$

$$B_{a\mu} \equiv \partial_\mu \varphi_a + A_{a\mu}$$

$$\lambda = \tau \frac{\partial n}{\partial \mu}$$

► KMS symmetry:

$$\mu(x) \rightarrow \mu(-x) \quad \varphi_a(x) \rightarrow - \left(\varphi_a + i\beta_0 (\mu - u^\mu A_\mu) \right) (-x)$$

$$B_{a\mu}(x) \rightarrow \left(B_{a\mu} + i\beta_0 (\partial_\mu \mu - E_\mu) \right) (-x)$$

$$\delta\Omega = n \delta\mu$$

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu \left(i\beta_0 u^\mu \Omega \right)$$

$$C_{\mu\nu}^{-1} = T_0 \sigma \Delta_{\mu\nu} - T_0 \lambda u^\mu u^\nu$$

► SK variables φ, φ_a

$$\mathcal{Z} = \int \mathcal{D}(\dots) e^{i \int_x \mathcal{L}} \text{ is unbounded}$$

$$\varphi(x) \rightarrow -\varphi(-x) \quad \varphi_a(x) \rightarrow - \left(\varphi_a + i \mathcal{E}_\beta \varphi \right) (-x)$$

$$\mu \equiv u^\mu (\partial_\mu \varphi + A_\mu) \quad \varphi \rightarrow \varphi + f(\mathbf{x})$$

Schwinger-Keldysh MIS Hydrodynamics

$$\mathcal{L} = n u^\mu B_{a\mu} + \mathcal{J}^\mu (B_{a\mu} - \psi_{a\mu}) - \sigma \Delta^{\mu\nu} \psi_{a\mu} \left(\partial_\nu \mu - E_\nu + \frac{\tau}{\sigma} u^\lambda \partial_\lambda \mathcal{J}_\nu \right) + i C_{\mu\nu}^{-1} \psi_a^\mu \psi_a^\nu$$

► Two KMS symmetries:

$$\mu(x) \rightarrow \mu(-x)$$

$$B_{a\mu}(x) \rightarrow \left(B_{a\mu} + i\beta_0 (\partial_\mu \mu - E_\mu) \right) (-x)$$

$$\mathcal{J}_\mu(x) \rightarrow \mathcal{J}_\mu(-x)$$

$$(a) \quad \psi_{a\mu}(x) \rightarrow \left(\psi_{a\mu} + i\beta_0 \Delta_\mu^\nu (\partial_\nu \mu - E_\nu) + i\beta_0 \frac{\tau}{\sigma} u^\lambda \partial_\lambda \mathcal{J}_\nu \right) (-x)$$

$$(b) \quad \psi_{a\mu}(x) \rightarrow - \left(\psi_{a\mu} + i\beta_0 \frac{1}{\sigma} \mathcal{J}_\nu \right) (-x)$$

$$\delta\Omega = n \delta\mu - \frac{1}{2} \frac{\tau}{\sigma} \delta\mathcal{J}^2$$

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu \left(i\beta_0 u^\mu \Omega \right)$$

$$C_{\mu\nu}^{-1} = T_0 \sigma \Delta_{\mu\nu}$$

Schwinger-Keldysh MIS Hydrodynamics

► Can recast the two KMS symmetries in terms of SK variables $\varphi, \varphi_a, v_\mu, v_{a\mu}$

► KMS symmetry (a):

$$\varphi(x) \rightarrow -\varphi(-x) \qquad \varphi_a(x) \rightarrow -\left(\varphi_a + i\mathfrak{L}_\beta \varphi\right)(-x)$$

$$v_\mu(x) \rightarrow v_\mu(-x) \qquad v_{a\mu}(x) \rightarrow \left(v_{a\mu} + i\mathfrak{L}_\beta v_\mu\right)(-x)$$

$$\mu \equiv u^\mu(\partial_\mu \varphi + A_\mu) \qquad \mathcal{J}_\mu \equiv v_\mu$$

$$\psi_{a\mu} \equiv \Delta_\mu^\nu B_{a\nu} + \frac{\tau}{\sigma} v_{a\mu}$$

$$\varphi \rightarrow \varphi + f(\mathbf{x})$$

► KMS symmetry (b):

$$\varphi(x) \rightarrow -\varphi(-x) \qquad \varphi_a(x) \rightarrow -\left(\varphi_a + i\mathfrak{L}_\beta \varphi\right)(-x)$$

$$v_\mu(x) \rightarrow -v_\mu(-x) \qquad v_{a\mu}(x) \rightarrow -\left(v_{a\mu} + i\mathfrak{L}_\beta v_\mu\right)(-x)$$

$$\mu \equiv u^\mu(\partial_\mu \varphi + A_\mu) \qquad \mathcal{J}_\mu \equiv u^\lambda \partial_\lambda v_\mu$$

$$\psi_{a\mu} \equiv \frac{1}{\sigma} v_{a\mu}$$

$$\varphi \rightarrow \varphi + f(\mathbf{x}) \qquad v_\mu \rightarrow v_\mu + f_\mu(\mathbf{x})$$

Schwinger-Keldysh BDNK* Hydrodynamics

- MIS hydrodynamics implies BDNK hydrodynamics when τ is constant

$$-\partial_\mu \left(\mathcal{J}^\mu + \tau \Delta^{\mu\nu} u^\lambda \partial_\lambda \mathcal{J}_\nu + \sigma \Delta^{\mu\nu} (\partial_\nu \mu - E_\nu) + \dots \right) = \partial_\mu \left(n u^\mu + \tau u^\mu u^\lambda \partial_\lambda n - \sigma \Delta^{\mu\nu} (\partial_\nu \mu - E_\nu) + \dots \right) = 0$$

But, $u^\mu A_\mu$ couples to n instead of $n + \tau u^\mu \partial_\mu n$.

- $\mathcal{J}_\mu$ can be integrated to set $\psi_{a\mu} = (1 - \tau u^\lambda \partial_\lambda)^{-1} B_{a\mu} \equiv \hat{B}_{a\mu}$

$$\hat{B}_{a\mu} = \partial_\mu \hat{\varphi}_a + \frac{1}{1 - \tau u^\lambda \partial_\lambda} A_{a\mu}$$

$$\hat{\varphi}_a = \frac{1}{1 - \tau u^\lambda \partial_\lambda} \varphi_a$$

$$\mathcal{L} = n u^\mu B_{a\mu} + \mathcal{J}^\mu (B_{a\mu} - \psi_{a\mu}) - \sigma \Delta^{\mu\nu} \psi_{a\mu} \left(\partial_\nu \mu - E_\nu + \frac{\tau}{\sigma} u^\lambda \partial_\lambda \mathcal{J}_\nu \right) + iT_0 \sigma \Delta_{\mu\nu} \psi_a^\mu \psi_a^\nu$$

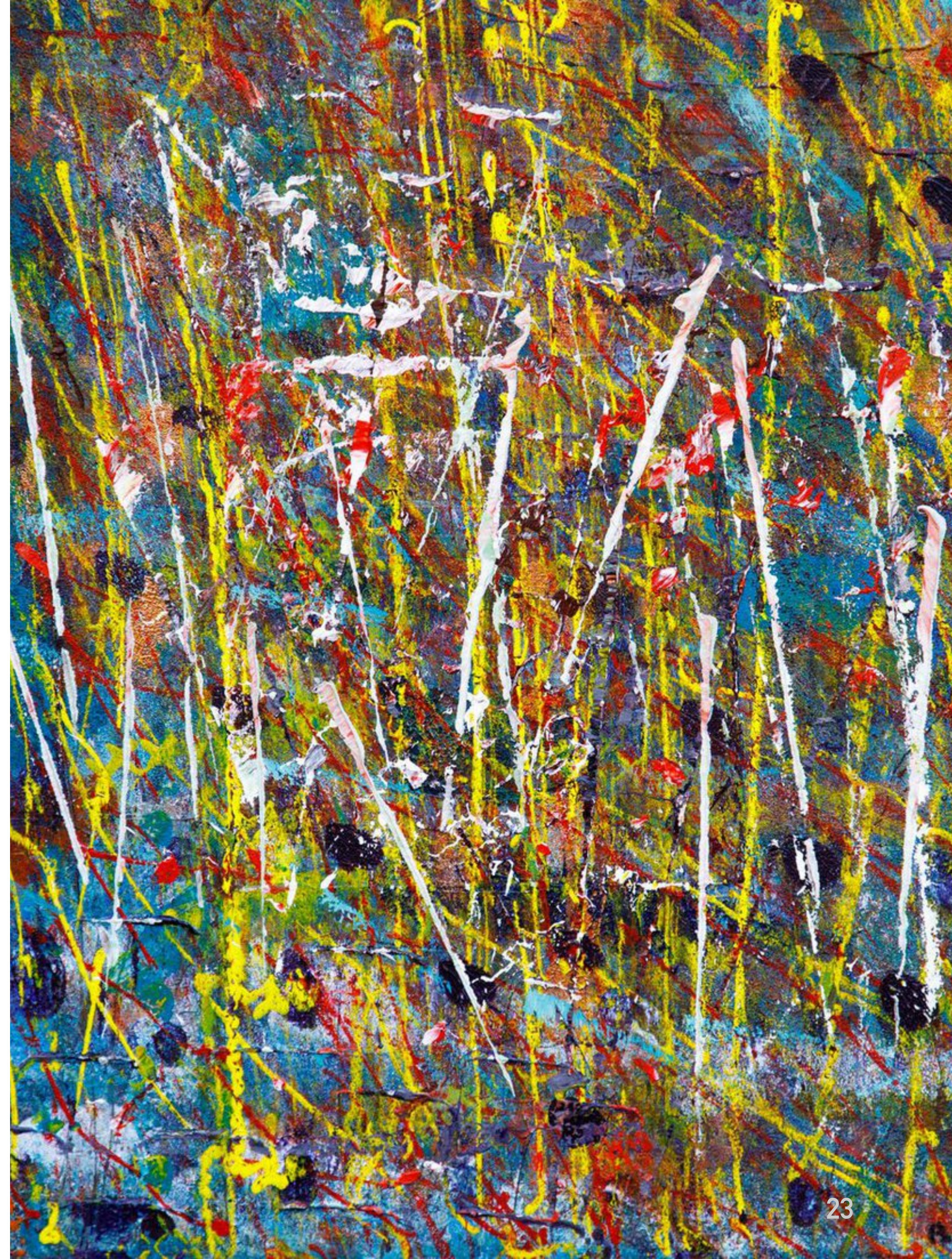
$$= \left(n u^\mu + \lambda u^\mu u^\nu \partial_\nu \mu \right) \hat{B}_{a\mu} - \sigma \Delta^{\mu\nu} \hat{B}_{a\mu} (\partial_\nu \mu - E_\nu) + iT_0 \sigma \Delta_{\mu\nu} \hat{B}_a^\mu \hat{B}_a^\nu - \cancel{iT_0 \lambda u^\mu u^\nu \hat{B}_a^\mu \hat{B}_a^\nu}$$

- This can be derived by imposing 2-point FDT.¹
Inconsistent with higher-point FDT when σ is not constant.²

$$\delta\Omega = n \delta\mu - \frac{1}{2} \frac{\tau}{\sigma} \delta \mathcal{J}^2$$

¹Abbasi, Kaminski, Tavakol [2212.11499], ²AJ, Kovtun [2309.00511]

OUTLOOK





Lessons

- Hydrodynamics is **not** completely characterised by its constitutive relations. New **stochastic transport coefficients**¹ “contaminate” hydrodynamic observables at higher derivative orders.
- Ensuring the 2-point FDT ($G^S \propto \text{Im } G^R$) in Martin-Siggia-Rose (MSR) formalism (stochastic hydrodynamics) **does not** guarantee consistency with higher-point FDTs.²
- Not all “stable and causal formalisms” of relativistic hydrodynamics are created equal: *Müller-Israel-Stewart (MIS) hydrodynamics* is consistent with stochastic fluctuations, while the newer *Bemfica-Disconzi-Noronha-Kovtun (BDNK) hydrodynamics*⁴ **is not**.^{2,3}
- Consistent UV-completions of hydrodynamics, like MIS hydrodynamics, may simultaneously realise **more than one** Kubo-Martin-Schwinger (KMS) symmetries.²

¹AJ, Kovtun [2009.01356]

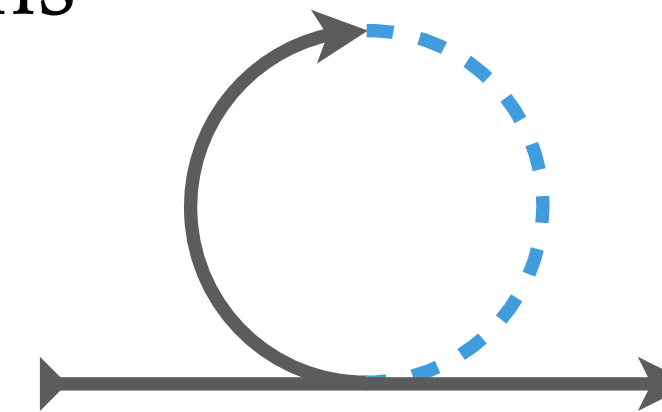
²AJ, Kovtun [2309.00511]

³Mullins, Hippert, Gavassino, Noronha [2309.00512]

⁴Kovtun [1907.08191], Bemfica, Disconzi, Noronha [1907.12695]

Frequency Regulators

- Bad (amputated) diagrams



- Retarded correlation functions are no longer purely-retarded

$$G^R = \longrightarrow \text{---} + \longrightarrow \text{---} \text{---} \text{---} + \longrightarrow \text{---} \text{---} \text{---}$$

- Need more powers of ω in the retarded propagator.
Need to introduce ghost fields.²
- Is there a ghost-free solution to the problem?
BDNK-like solutions¹ give sign-problems in $\text{Im } \mathcal{L}$.
MIS-like solutions do not work.

¹Gao, Glorioso, Liu [1803.10778]

²Gao, Liu [1701.07445], Jensen, Pinzani-Fokeeva, Yarom [1701.07436]



Extensions of Schwinger-Keldysh

- Relativistic magnetohydrodynamics

[Glorioso, Son [1811.04879]] [Gralla, Iqbal [1811.04879]]

- Spontaneously broken symmetries

- $U(1)$ — superfluids

[Donos, Kailidis [2304.06008]]
[Baggioli, Bu, Ziogas [2304.14173]]

- Translations — crystals and elasticity

[Baggioli, Landry [2008.05339]]
[Baggioli, Landry, Zaccone [2101.05015]]

- Rotations — Nematic liquid crystals, etc.

[Landry [2309.15142]]

- Hydrodynamics with exotic symmetries

- Dipole, multipole, and subsystem symmetries (fractons)

- Higher-form symmetries [Glorioso, Son [1811.04879]] [Gralla, Iqbal [1811.04879]]

- Higher-group and non-invertible symmetries

[Das, Iqbal, Poovuttikul [2212.09787]]

- Non-relativistic hydrodynamics

[AJ [2008.03994]] [AJ, Armas [2010.15782]]



Thank you

References

AJ, P Kovtun [2309.00511].

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