

A field theorist's take on fluid dynamics

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Motivation

- ▶ Our understanding of zero temperature quantum field theories is quite mature. However, we still do not have a reasonable control over finite temperature field theories.
- ▶ Equilibrium phenomenon in finite temperature quantum field theories can be described by performing a Wick rotation and using the Euclidean field theoretic framework.
- ▶ Out of equilibrium phenomenon in thermal quantum field theories are still an active topic of research.
- ▶ Hydrodynamics attempts to describe out of equilibrium effects in thermal quantum field theories, in the regime where the deviations away from equilibrium are sufficiently small.
- ▶ More often than not, when we are interested in a physical thermal system, we have little or no knowledge of the underlying microscopic field content, let alone an action principle.
- ▶ Hydrodynamics aims at being a universal description, with minimal assumptions about the underlying microscopic field content.

Defining fluids

- ▶ A fluid is a anything that flows and takes the shape of the container it is kept in.
— My school textbook
- ▶ Fluid is a substance that flows and is not solid.
— Cambridge dictionary
- ▶ A fluid is a substance that continually deforms under an applied shear.
— Wikipedia
- ▶ Fluids are the universal long-wavelength effective description of near-equilibrium dynamics of interacting quantum systems.
— [Haehl, Loganayagam, Rangamani '15]
- ▶ Fluids are an attempt at describing low energy fluctuations of a thermal quantum system around its thermodynamic equilibrium, in absence of the knowledge of an underlying microscopic field theory, guided by symmetries and other empirical physical expectations.
— My understanding

Symmetries and Noether theorem

- ▶ Noether theorem guarantees that any theory invariant under spacetime Poincaré and internal U(1) transformations, contains an associated *energy-momentum tensor* $T^{\mu\nu}$ and *charge current* J^μ , which are conserved onshell

$$\partial_\mu T^{\mu\nu} = c_i^\mu \mathcal{E}_i \simeq 0, \quad \partial_\mu J^\mu = c_i \mathcal{E}_i \simeq 0.$$

\mathcal{E}_i : equations of motion, c_i^μ, c_i : arbitrary multipliers.

- ▶ For example, consider a free complex scalar field theory

$$S = \int d^4x \left(\partial_\mu \psi^\dagger \partial^\mu \psi + m^2 \psi^\dagger \psi \right),$$

with equations of motion

$$\mathcal{E} = \left(\partial_\mu \partial^\mu \psi - m^2 \psi \right) \simeq 0, \quad \mathcal{E}^\dagger = \left(\partial_\mu \partial^\mu \psi^\dagger - m^2 \psi^\dagger \right) \simeq 0.$$

The energy-momentum tensor and charge current are given as

$$T^{\mu\nu} = \left(\partial_\rho \psi^\dagger \partial^\rho \psi + m^2 \psi^\dagger \psi \right) \eta^{\mu\nu} - 2\partial^{(\mu} \psi^\dagger \partial^{\nu)} \psi, \quad J^\mu = i\psi^\dagger \partial^\mu \psi - i\psi \partial^\mu \psi^\dagger,$$

with conservation laws

$$\partial_\mu T^{\mu\nu} = -\partial^\nu \psi^\dagger \mathcal{E} - \partial^\nu \psi \mathcal{E}^\dagger \simeq 0, \quad \partial_\mu J^\mu = i\psi^\dagger \mathcal{E}_i - i\psi \mathcal{E}_i^\dagger \simeq 0.$$

Hydrodynamic fields

- ▶ We can use the conservation laws as a placeholder for some equations of motion

$$\partial_\mu T^{\mu\nu} \simeq 0, \quad \partial_\mu J^\mu \simeq 0.$$

- ▶ In a low energy regime, when we know nothing about the UV complete field theory, we are still guaranteed to have a $T^{\mu\nu}$ and J^μ , whose conservation laws will serve as equations of motion for low energy effective degrees of freedom.
- ▶ We identify these low energy effective degrees of freedom as

$$u^\mu : \text{Velocity } (u^\mu u_\mu = -1), \quad T : \text{Temperature}, \quad \mu : \text{Chemical potential.}$$

They are called the *hydrodynamic fields*.

- ▶ Hydrodynamic fields can admit arbitrary redefinition among themselves.
- ▶ The choice of fields is motivated by the thermal density matrix in equilibrium

$$\hat{\rho} = \frac{1}{Z} \exp\left(\frac{1}{T} u_\mu \hat{P}^\mu + \frac{\mu}{T} \hat{Q}\right), \quad Z = \text{tr} \left[\exp\left(\frac{1}{T} u_\mu \hat{P}^\mu + \frac{\mu}{T} \hat{Q}\right) \right].$$

- ▶ They can also be understood as Goldstone modes of the Poincaré and U(1) invariance broken by thermal excitations.

Hydrodynamics regime

Hydrodynamic description assumes that

- ▶ There is a mass gap in the theory: no excitations of arbitrarily small energies.
- ▶ Symmetries of the microscopic theory are not broken in the fluid phase, as this would lead to Goldstone modes that need no energy to excite.
- ▶ Thermal fluctuations have energies lower than the mass gap, so that none of the fundamental fields are excited at hydrodynamic energy scales. In other words, fluctuations are on the length scales larger than the mean-free path of the system.

In practise it implies that

- ▶ u^μ , T and μ are the only fields of interest in the hydrodynamic regime.
- ▶ Derivatives of u^μ , T and μ are small and can be treated as a perturbative parameter.
- ▶ For a generic fluid, $T^{\mu\nu}$ and J^μ can be expressed purely in terms of u^μ , T , μ and their derivatives, arranged as a perturbative expansion in derivatives. These expressions are called *constitutive relations* of the fluid.

Constitutive relations

- ▶ At a given derivative order, $T^{\mu\nu}$ and J^μ will admit all the possible tensor structures made out of u^μ , T , μ and their derivatives, multiplied with arbitrary functions of T and μ . These multipliers are called the *transport coefficients* of the fluid.
- ▶ Truncated to a finite derivative order, a fluid is completely characterised by a finite set of transport coefficients. These transport coefficients are specific to the fluid in question and depends on the specifics of the microscopic theory.
- ▶ These unknown transport coefficients are free parameters of the fluid theory. They can either be fixed by experiments, or if the microscopic theory in question is known, they can be explicitly computed.
- ▶ Even without the knowledge of the microscopic theory however, we can impose some strict constraints on the transport coefficients by imposing some physicality requirements, such as the second law of thermodynamics.

Ideal fluid constitutive relations

- ▶ For example, let us consider fluid dynamics at zero derivative order. These are commonly known as ideal fluids.

$$T^{\mu\nu} = \epsilon(T, \mu) u^\mu u^\nu + P(T, \mu) P^{\mu\nu} + \mathcal{O}(\partial), \quad J^\mu = q(T, \mu) u^\mu + \mathcal{O}(\partial).$$

$P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$ is the projection operator transverse to u^μ .

- ▶ Transport coefficients can be identified as

$\epsilon(T, \mu)$: Energy density, $P(T, \mu)$: Pressure, $q(T, \mu)$: Charge density.

- ▶ Conservation laws imply the first derivative equations of motion

$$\partial_\mu T^{\mu\nu} = u^\nu \left[\underbrace{u^\mu \partial_\mu \epsilon + (\epsilon + P) \partial_\mu u^\mu}_{T\text{-EOM}} \right] + \left[\underbrace{(\epsilon + P) u^\mu \partial_\mu u^\nu + P^{\mu\nu} \partial_\mu P}_{u^\mu\text{-EOM}} \right] + \mathcal{O}(\partial^2) \simeq 0,$$

$$\partial_\mu J^\mu = \left[\underbrace{u^\mu \partial_\mu q + q \partial_\mu u^\mu}_{\mu\text{-EOM}} \right] + \mathcal{O}(\partial^2) \simeq 0.$$

- ▶ For a given fluid with known $\epsilon(T, \mu)$, $P(T, \mu)$ and $q(T, \mu)$, these equations of motion can be used to solve the dynamics of hydrodynamic fields.

Second law of thermodynamics

Though we cannot give the explicit functional form of these transport coefficients, we can impose some strict constraints on them based on certain empirical physicality requirements.

The most useful of these requirements is the *second law of thermodynamics*.

Thermodynamically isolated fluid configurations

A fluid configuration is said to be thermodynamically isolated, if it satisfies the conservation laws.

Local second law of thermodynamics

Given a set of constitutive relations $T^{\mu\nu}$ and J^μ , there exists an entropy current J_S^μ whose divergence is non-negative for all *thermodynamically isolated* fluid configurations,

$$\partial_\mu J_S^\mu \gtrsim 0.$$

A field theoretic interpretation of this law is still an open problem.

Offshell local second law of thermodynamics

Offshell local second law of thermodynamics

Given a set of constitutive relations $T^{\mu\nu}$ and J^μ , there exists an *entropy current* J_S^μ whose divergence is non-negative up to arbitrary combinations of the conservation laws,

$$\partial_\mu J_S^\mu + \beta_\nu \partial_\mu T^{\mu\nu} + \Lambda_\beta \partial_\mu J^\mu \geq 0.$$

β_ν and Λ_β are some arbitrary Lagrange multipliers.

If we define $N^\mu = J_S^\mu + T^{\mu\nu} \beta_\nu + J^\mu \Lambda_\beta$, this law can be rewritten as

Offshell local second law of thermodynamics

Given a set of constitutive relations $T^{\mu\nu}$ and J^μ , there exists a *free energy current* N^μ which satisfies

$$\partial_\mu N^\mu - T^{\mu\nu} \partial_\mu \beta_\nu - J^\mu \partial_\mu \Lambda_\beta \geq 0.$$

β_ν and Λ_β are some arbitrary Lagrange multipliers.

Remember the field redefinition freedom in u^μ , T and μ ? We can fix it by choosing the Lagrange multipliers $\beta^\mu = u^\mu / T$, $\Lambda_\beta = \mu / T$.

Second law at ideal order

- ▶ Recall the ideal order constitutive relations

$$T^{\mu\nu} = \epsilon(T, \mu) u^\mu u^\nu + P(T, \mu) P^{\mu\nu} + \mathcal{O}(\partial), \quad J^\mu = q(T, \mu) u^\mu + \mathcal{O}(\partial).$$

Consider the most generic free energy current at zero derivative order

$$N^\mu = N(T) u^\mu + \mathcal{O}(\partial).$$

- ▶ Offshell second law of thermodynamics implies that

$$\partial_\mu N^\mu - T^{\mu\nu} \partial_\mu \frac{u_\nu}{T} - J^\mu \partial_\mu \left(\frac{\mu}{T} \right) = \left(N - \frac{P}{T} \right) \partial_\mu u^\mu + u^\mu \left(\partial_\mu N - \frac{\epsilon}{T^2} \partial_\mu T - q \partial_\mu \left(\frac{\mu}{T} \right) \right) \geq 0.$$

- ▶ We can read out that $N = P/T$ and $\delta N = \frac{\epsilon}{T^2} \delta T - q \delta \left(\frac{\mu}{T} \right)$. Eliminating N

$$\delta P = \frac{\epsilon + P - \mu q}{T} \delta T + q \delta \mu \quad \implies \quad \delta \epsilon = T \delta \left(\frac{\epsilon + P - \mu q}{T} \right) + \mu \delta q$$

- ▶ Defining an entropy density $s = (\epsilon + P - \mu q)/T$, this gives us the thermodynamics

First law: $\delta \epsilon = T \delta s + \mu \delta q,$	Euler equation $\epsilon + P = T s + \mu q.$
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Aside: first law of thermodynamics and Euler equation

- ▶ All of us know the first law of thermodynamics

$$\delta E = T\delta S - P\delta V + \mu\delta Q.$$

- ▶ A lesser known thermodynamic statement is the Euler equation. It follows from the assumption that energy scales homogeneously

$$\lambda E(S, V, Q) = E(\lambda S, \lambda V, \lambda Q).$$

Differentiating this equation with respect to λ and setting $\lambda = 1$ we find

$$E = TS - PV + \mu Q.$$

- ▶ Defining densities $\epsilon = E/V$, $s = S/V$ and $q = Q/V$, they can be expressed as

$$\delta\epsilon = \frac{1}{V}\delta E - \frac{E}{V^2}\delta V = \frac{T}{V}\delta S + \frac{\mu}{V}\delta Q - \frac{E + PV}{V^2}\delta V = T\delta s + \mu\delta q \implies \delta\epsilon = T\delta s + \mu\delta q,$$

$$\epsilon = \frac{E}{V} = \frac{TS - PV + \mu Q}{V} = T s - P + \mu q \implies \epsilon + P = T s + \mu q.$$

These are the local versions of first law of thermodynamics and Euler equation.

Recap: ideal fluids

Ideal fluids

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P P^{\mu\nu} + \mathcal{O}(\partial), \quad J^\mu = q u^\mu + \mathcal{O}(\partial), \quad J_S^\mu = s u^\mu + \mathcal{O}(\partial).$$

Transport coefficients:

ϵ : Energy density, P : Pressure, q : Charge density, s : Entropy density.

Constraints:

$$\delta\epsilon = T\delta s + \mu\delta q \quad (\text{First law}), \quad \epsilon + P = T s + \mu q \quad (\text{Euler equation}).$$

- Note that a combination of the first law and Euler relation gives the *Gibbs-Duhem equation*

$$\delta P = s\delta T + q\delta\mu.$$

It follows that

$$s = \frac{\partial P}{\partial T}, \quad q = \frac{\partial P}{\partial \mu}, \quad \epsilon = T \frac{\partial P}{\partial T} + \mu \frac{\partial P}{\partial \mu} - P.$$

- Consequently, an ideal fluid is completely characterised by a function $P(T, \mu)$ known as the *equation of state* of the fluid.

Recap: strategy of hydrodynamics

- ▶ Depending on the underlying symmetries, we start with certain *conservation laws*, e.g. energy momentum conservation $\partial_\mu T^{\mu\nu}$ and charge conservation $\partial_\mu J^\mu$.
- ▶ We treat the conservation laws as *equations of motion* for some arbitrarily chosen *hydrodynamic fields*: four-velocity u^μ , temperature T and chemical potential μ .
- ▶ We write the most generic expressions for $T^{\mu\nu}$ and J^μ in terms of u^μ , T , μ and their derivatives, called the *constitutive relations*. These constitutive relations can be arranged order by order in a *derivative expansion*.
- ▶ At n th derivative order, constitutive relations contain all possible tensor structures made out of u^μ , T , μ and n derivatives, multiplied with arbitrary functions of T and μ called *transport coefficients*.
- ▶ We impose some physicality requirements on these constitutive relations, which give various constraints on the transport coefficients. Most important of these requirements is the *second law of thermodynamics*, which demands the existence of a *free energy current* N^μ which satisfies $\partial_\mu N^\mu - T^{\mu\nu} \partial_\mu \left(\frac{u_\nu}{T} \right) - J^\mu \partial_\mu \left(\frac{\mu}{T} \right) \geq 0$.
- ▶ We can impose some additional physicality requirements such as CPT invariance and microscopic reversibility (Onsager relations).

First order fluid constitutive relations

- ▶ First derivatives of u^μ and T are given as

$$\partial_\mu u^\nu, \quad \partial_\mu T, \quad \partial_\mu \left(\frac{\mu}{T} \right).$$

- ▶ We can decompose them into

$$\text{Scalars:} \quad \Theta = \partial_\mu u^\mu, \quad u^\mu \partial_\mu T, \quad u^\mu \partial_\mu \left(\frac{\mu}{T} \right),$$

$$\text{Vectors:} \quad u^\nu \partial_\nu u^\mu, \quad P^{\mu\nu} \partial_\nu T, \quad P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right),$$

$$\text{Symmetric tensors:} \quad \sigma^{\mu\nu} = P^{\mu\rho} P^{\nu\sigma} \left(\partial_\rho u_\sigma + \partial_\sigma u_\rho - \frac{2}{3} P_{\rho\sigma} \partial_\tau u^\tau \right),$$

$$\text{Antisymmetric tensors:} \quad \omega^{\mu\nu} = P^{\mu\rho} P^{\nu\sigma} (\partial_\rho u_\sigma - \partial_\sigma u_\rho).$$

Entries in red have been eliminated by the first order equations of motion.

- ▶ Constitutive relations can be given as

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P P^{\mu\nu} + a_1 \Theta u^\mu u^\nu + 2u^{(\mu} \left(\alpha_2 P^{\nu)\rho} \partial_\rho T + \alpha_3 P^{\nu)\rho} \partial_\rho \left(\frac{\mu}{T} \right) \right) \\ - \eta \sigma^{\mu\nu} - \zeta \Theta P^{\mu\nu} + \mathcal{O}(\partial^2).$$

$$J^\mu = q u^\mu + \alpha_4 \Theta u^\mu + \alpha_5 P^{\mu\nu} \partial_\nu T - \sigma T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right).$$

Second law at first order

We start with the most generic free energy current possible

$$N^\mu = \frac{1}{T} P u^\mu + \beta_1 \Theta u^\mu + \beta_2 P^{\mu\nu} \partial_\nu T + \beta_3 T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) + \mathcal{O}(\partial).$$

The second law implies

$$\begin{aligned} & \partial_\mu N^\mu - T^{\mu\nu} \partial_\mu \left(\frac{u_\nu}{T} \right) - J^\mu \partial_\mu \left(\frac{\mu}{T} \right) \\ &= \partial_\mu (\beta_1 \Theta u^\mu) + \partial_\mu (\beta_2 P^{\mu\nu} \partial_\nu T) + \partial_\mu \left(\beta_3 T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) \right) + \frac{1}{2T} \eta \sigma^{\mu\nu} \sigma_{\mu\nu} + \frac{1}{T} \zeta \Theta^2 \\ & \quad - \frac{\alpha_1}{T^2} \Theta u^\mu \partial_\mu T - \frac{1}{T} \left(\frac{1}{T} \partial_\mu T + u^\nu \partial_\nu u_\mu \right) \left(\alpha_2 P^{\mu\rho} \partial_\rho T + \alpha_3 P^{\mu\rho} \partial_\rho \left(\frac{\mu}{T} \right) \right) \\ & \quad - \alpha_4 \Theta u^\mu \partial_\mu \left(\frac{\mu}{T} \right) - \alpha_5 P^{\mu\nu} \partial_\mu \left(\frac{\mu}{T} \right) \partial_\nu T + \sigma T P^{\mu\nu} \partial_\mu \left(\frac{\mu}{T} \right) \partial_\nu \left(\frac{\mu}{T} \right) \geq 0. \end{aligned}$$

Non-negativity implies the constraints

$$\beta_1 = \beta_2 = \beta_3 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \eta \geq 0, \quad \zeta \geq 0, \quad \sigma \geq 0.$$

Recap: first order fluid

First order fluid constitutive relations

Constitutive relations:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P P^{\mu\nu} - \eta \sigma^{\mu\nu} - \zeta \Theta P^{\mu\nu} + \mathcal{O}(\partial^2), \quad J^\mu = q u^\mu - \sigma T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) + \mathcal{O}(\partial^2).$$

Free energy and entropy currents:

$$N^\mu = \frac{1}{T} P u^\mu + \mathcal{O}(\partial^2), \quad J_S^\mu = N^\mu - T^{\mu\nu} \frac{u_\nu}{T} - J^\mu \frac{\mu}{T} = s u^\mu + \sigma \mu P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) + \mathcal{O}(\partial^2).$$

Transport coefficients:

$$\begin{aligned} \epsilon &: \text{Energy density}, & P &: \text{Pressure}, & s &: \text{Entropy density}, \\ \eta &: \text{Shear viscosity}, & \zeta &: \text{Bulk viscosity}, & \sigma &: \text{Conductivity}. \end{aligned}$$

Constraints:

$$\delta\epsilon = T\delta s + \mu\delta q, \quad \epsilon + P = T s + \mu q, \quad \eta \geq 0, \quad \zeta \geq 0, \quad \sigma \geq 0.$$

Generalisations

- ▶ First obvious generalisation is to higher derivative orders. Second order analysis has been explicitly done [Romatschke '09; Bhattacharyya '12]. We also know an algorithm to recover constraints from the second law of thermodynamics at arbitrarily high derivative orders [Bhattacharyya '13].
- ▶ This discussion can also be generalised to non-Abelian internal symmetries. Spacetime or internal symmetries could also be anomalous [Haehl et al.'13].
- ▶ We can impose additional physicality requirements, such as CPT invariance, microscopic time reversal invariance (Onsager relations) and existence of a partition function in equilibrium [Banerjee et al.; Jensen et al.'12].
- ▶ We can couple the fluid to a slowly varying background metric and gauge field.
- ▶ We can replace the Poincaré group with the Galilean group to obtain non-relativistic fluids. There is also a *null fluid framework* which allows us to study non-relativistic fluids directly using relativistic fluids [Banerjee, Dutta, AJ '15].
- ▶ We can also generalise hydrodynamics to spontaneously broken spacetime or internal symmetries. [Bhattacharya et al.'11; Armas et al.'15; AJ '16].

Hydrodynamics with gapless modes

- ▶ In our discussion of hydrodynamics, we had assumed that the theory does not contain any gapless modes. This was to justify the assumption that u^μ , T and μ are the only relevant fields in the constitutive relations.
- ▶ It is possible to do hydrodynamics with additional gapless modes, but the structure of the constitutive relations crucially depends on how the additional modes transform under symmetries of the theory.
- ▶ All the original transport coefficients are still present, but now there will be many more coefficients which capture the response of energy-momentum tensor and charge current to fluctuations of the additional mode.
- ▶ We still need to assume that the fluctuations of the additional modes lie in the hydrodynamic regime, i.e. their derivatives are small.
- ▶ A particularly interesting case is of the Goldstone modes coming from spontaneously broken symmetries in the fluid phase.

Setup of superfluid dynamics

- ▶ Let us consider hydrodynamics with spontaneously broken internal U(1) symmetry. The associated Goldstone mode is a U(1) phase φ which transforms as

$$\varphi \rightarrow \varphi - \Lambda.$$

- ▶ Let us denote the equation of motion of φ by a placeholder

$$K \approx 0.$$

- ▶ It can be checked that offshell configurations of φ modify the conservation laws as

$$\partial_\mu T^{\mu\nu} - K \partial^\mu \varphi \simeq 0, \quad \partial_\mu J^\mu + K \simeq 0.$$

- ▶ Low energy effective modes contain u^μ , T , μ and in addition

$$\xi_\mu = \partial_\mu \varphi : \quad \text{Superfluid velocity.}$$

There is field redefinition invariance in u^μ , T and μ , but the ξ_μ is uniquely defined in terms of the microscopic Goldstone boson.

- ▶ ξ_μ is taken to be order zero in the derivative expansion.

Second law of thermodynamics

We need to find the most generic superfluid constitutive relations $T^{\mu\nu}$, J^μ and K in terms of u^μ , T , μ and ξ_μ . They satisfy the local second law of thermodynamics:

Local second law of thermodynamics

Given a set of constitutive relations $T^{\mu\nu}$, J^μ and K , there exists an entropy current J_S^μ whose divergence is non-negative for all *thermodynamically isolated* superfluid configurations,

$$\partial_\mu J_S^\mu \gtrsim 0.$$

Offshell local second law of thermodynamics

Given a set of constitutive relations $T^{\mu\nu}$, J^μ and K , there exists a *free energy current* N^μ which satisfies,

$$\partial_\mu N^\mu - T^{\mu\nu} \partial_\mu \beta_\nu - J^\mu \partial_\mu \Lambda_\beta - K (\beta^\mu \xi_\mu - \Lambda_\beta).$$

β_ν and Λ_β are some arbitrary Lagrange multipliers.

Entropy and free energy currents are related via $N^\mu = J_S^\mu + T^{\mu\nu} \beta_\nu + J^\mu \Lambda_\beta$.

Hydrodynamic redefinition freedom can be fixed by choosing $\beta^\mu = u^\mu / T$, $\Lambda_\beta = \mu / T$.

Josephson equation

- ▶ Let us look at the offshell second law little carefully

$$\partial_\mu N^\mu - T^{\mu\nu} \partial_\mu \left(\frac{u_\nu}{T} \right) - J^\mu \partial_\mu \left(\frac{\mu}{T} \right) - \frac{1}{T} K (u^\mu \xi_\mu - \mu) \geq 0.$$

At zero derivative order, this truncates to

$$-\frac{1}{T} K (u^\mu \xi_\mu - \mu) + \mathcal{O}(\partial) \geq 0.$$

- ▶ Therefore, at zero derivative order, K is simply given by

$$K = -\alpha (u^\mu \xi_\mu - \mu) + \mathcal{O}(\partial), \quad \alpha \geq 0.$$

- ▶ The equation of motion for the Goldstone mode φ is given by $K \approx 0$,

$$u^\mu \xi_\mu = u^\mu \partial_\mu \varphi \approx \mu + \mathcal{O}(\partial).$$

This is known as the *Josephson equation*.

- ▶ We have derived the Josephson equation using the second law of thermodynamics.

Second law constraints on Ideal superfluids

- ▶ We start with the most generic constitutive relations

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + PP^{\mu\nu} + \rho_s \zeta^\mu \zeta^\nu + 2\epsilon_s u^{(\mu} \zeta^{\nu)} + \mathcal{O}(\partial), \quad J^\mu = qu^\mu + q_s \zeta^\mu + \mathcal{O}(\partial).$$

$$K = -\alpha(u^\mu \zeta_\mu - \mu) + \mathcal{K} + \mathcal{O}(\partial^2).$$

- ▶ For the free energy current we have

$$N^\mu = Nu^\mu + N_s \zeta^\mu.$$

- ▶ The second law gives

$$\begin{aligned} & \partial_\mu N^\mu - T^{\mu\nu} \partial_\mu \left(\frac{u_\nu}{T} \right) - J^\mu \partial_\mu \left(\frac{\mu}{T} \right) - \frac{1}{T} K (u^\mu \zeta_\mu - \mu) \\ &= \left(N - \frac{P}{T} \right) \partial_\mu u^\mu + u^\mu \left(\partial_\mu N - \frac{\epsilon}{T^2} \partial_\mu T - q \partial_\mu \left(\frac{\mu}{T} \right) + \frac{1}{2T} \rho_s \partial_\mu \zeta^2 \right) \\ & \quad - (\rho_s + q_s) \zeta^\mu \partial_\mu \left(\frac{\mu}{T} \right) + \nabla_\mu \left(N_s \zeta^\mu - \frac{1}{T} (u^\nu \zeta_\nu - \mu) \rho_s \zeta^\mu \right) \\ & \quad + \frac{1}{T} \alpha (u^\mu \zeta_\mu - \mu)^2 - \frac{1}{T} (\mathcal{K} - \partial_\nu (\rho_s \zeta^\nu)) (u^\mu \zeta_\mu - \mu) - 2\epsilon_s u^{(\mu} \zeta^{\nu)} \partial_\mu \left(\frac{u_\nu}{T} \right) \gtrsim 0. \end{aligned}$$

- ▶ Constraints:

$$\begin{aligned} N &= \frac{P}{T}, & N_s &= \frac{1}{T} (u^\nu \zeta_\nu - \mu) \rho_s, & \delta P &= \frac{\epsilon + P - \mu q}{T} \delta T + q \delta \mu - \frac{1}{2} \rho_s \delta \zeta^2, \\ \epsilon_s &= 0, & q_s &= -\rho_s, & \mathcal{K} &= \partial_\nu (\rho_s \zeta^\nu). \end{aligned}$$

Recap: ideal superfluids

Ideal superfluid constitutive relations

Constitutive relations:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P P^{\mu\nu} + \rho_s \xi^\mu \xi^\nu + \mathcal{O}(\partial), \quad J^\mu = q u^\mu - \rho_s \xi^\mu + \mathcal{O}(\partial).$$

Josephson equation (φ equation of motion):

$$u^\mu \xi_\mu + u^\mu \partial_\mu \varphi \simeq \mu + \frac{1}{\alpha} \partial_\nu (\rho_s \xi^\nu) + \mathcal{O}(\partial^2).$$

Free energy and entropy currents:

$$N^\mu = \frac{1}{T} P u^\mu + \frac{1}{T} (u^\nu \xi_\nu - \mu) \rho_s \xi^\mu, \quad J_S^\mu = N^\mu - \frac{1}{T} T^{\mu\nu} u_\nu - \frac{1}{T} J^\mu \mu = s u^\mu.$$

Transport coefficients:

ϵ : Energy density, P : Pressure, ρ_s : Superfluid density, s : Entropy density.

Constraints:

$$\delta\epsilon = T\delta s + \mu\delta q + \frac{1}{2}\rho_s\delta\xi^2, \quad \epsilon + P = T s + \mu q, \quad \alpha \geq 0.$$

- ▶ This prescription can be generalised to higher derivative orders. First order superfluids has been looked at in full generality [Bhattacharya et al.'11]. No one has attempted second order superfluids. However, we have a generic algorithm to find second law constraints on superfluids at arbitrary derivative orders [AJ '16].
- ▶ We can spontaneously break non-Abelian internal symmetries [AJ '16]. These symmetries can also be anomalous [Bhattacharya et al.'11].
- ▶ Instead of breaking internal symmetries, we can also spontaneously break spacetime symmetries. This could lead to formation of surfaces in fluids [Armas et al.'15]. Breaking of translations has also been used to study momentum relaxation in hydrodynamics [Andrade, Withers '13].
- ▶ We can switch the Poincaré with the Galilean group to study non-relativistic superfluids [Banerjee, Bhatkar, AJ '17].

- ▶ In absence of a concrete framework to deal with out of equilibrium quantum field theories, hydrodynamics can help us deal with perturbative deviations away from thermodynamic equilibrium.
- ▶ Hydrodynamics is universal, i.e. it does not rely on the microscopic details of the theory. Only ingredients that go in are the symmetries and empirical physicality requirements such as the second law of thermodynamics.
- ▶ Hydrodynamics completely determines a fluid up to some transport coefficients. These coefficients are characteristic of the fluid in question and depend on the microscopic field theory.
- ▶ We can impose some strict constraints on these transport coefficients by requiring the second law of thermodynamics to hold. We can also impose other physicality requirements such as CPT invariance and microscopic reversibility.
- ▶ Hydrodynamics can be extended to deal with spontaneously broken symmetries. This leads to extra transport coefficients as opposed to ordinary fluids.
- ▶ Equations of motion of the Goldstone modes follows from within the framework of hydrodynamics.

Further directions

- ▶ Hydrodynamics leaves some transport coefficients undermined, which characterise different fluids. For fluids with a holographic dual, these transport coefficients can be computed using the fluid/gravity correspondence [Bhattacharyya et al.'07].
- ▶ Ordinary fluid transport coefficients to second order [Bhattacharyya et al.'07] and superfluid transport coefficients to first order [Bhattacharya et al.'11] have been computed holographically.
- ▶ Hydrodynamics is advertised as a low-energy effective field theory. It would be interesting to see if the hydrodynamic setup can be derived from a low energy Wilsonian effective action.
- ▶ We now have a Wilsonian effective action for adiabatic (non-dissipative) hydrodynamics [Haehl et al.'13]. Limited progress for dissipative hydrodynamics has also been made [Haehl et al.'15].

Thank You