HOLOGRAPHIC FERROMAGNETISM

AND

NON-RELATIVISTIC CHARGED HYDRODYNAMICS

A REPORT

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in

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by

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Preface

This text is a brief report on the work I have been involved in as my master's final year thesis project, carried out at *Indian Institute of Science Education and Research (IISER), Bhopal* from May 2013 to April 2014, under the supervision of *Dr. Suvankar Dutta*, Assistant Professor, Dept. of Physics, IISER Bhopal.

The report is divided into two parts: *Holographic Ferromagnetism* and *Non-relativistic Charged Hydrodynamics*. The first part (Holographic Ferromagnetism) concerns the study of ferromagnetic properties of a conformal field theory in the context of the AdS/CFT conjecture in String Theory. The work has been published in the *Journal of High Energy Physics* (JHEP) [1]. In this report we present the main idea of this work and physical aspects of our results. A rigorous and technical discussion can be found in the aforementioned research article.

In Part-II (Non-relativistic Charged Hydrodynamics) we try to understand different transport properties of a parity-odd, non-relativistic charged fluid in presence of background electro-magnetic fields. Especially, we construct a consistent entropy current for the non-relativistic fluid and impose constraints on the various transport coefficients from the positivity of local entropy production. A preprint of this work can be found on arXiv [2]. Here again, most of the technical issues have been omitted to keep the report accessible to a broader audience.

Interested readers can jump to either of the parts directly, as none of them impair the readability of other. However within a part, it is suggestive to go through the *'Introduction and Background'* section to begin with. Each part has a *'Discussion'* section where we summarise our main results and scopes for further analysis. The report has six appendices, four belonging to Part-I and two to Part-II. There we discuss some of the related but fairly sidelined topics from the main text.

I hope the readers will find the report quite interesting and fairly accessible. Any queries or clarifications can be directed to ajainphysics@gmail.com.

Akash Jain
April 2014, IISER Bhopal

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First and foremost, I would like to present my sincere gratitude towards my MS thesis supervisor, *Dr. Suvankar Dutta* for his constant support and motivation throughout the project.

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I find myself fortunate to have an opportunity to work under such friendly supervision, and am grateful to the unparalleled efforts my supervisor and collaborators for making my first research experience so wonderful.

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I would like to thank INSPIRE, Department of Science and Technology, MHRD, Govt. of India for their generous funding throughout my graduation. I am also grateful to the Department of Physics, IISER Bhopal for their support and providing me with the required infrastructure and facilities to successfully complete my graduation and carry out this project.

Finally I would like to thank all my friends, family members and my parents for their moral support and motivation, which enabled me to concentrate better towards my academia.

Abstract

In the first part of this thesis, we study thermodynamic and magnetic properties of a conformal field theory living on the surface of a two-sphere using the AdS/CFT correspondence. The correspondence is an efficient tool to study properties of a strongly coupled quantum field theory in terms of its weakly coupled super-gravity dual. We find that our field theory exhibits a ferromagnetic-type phase transition. At high temperature and in absence of external magnetic field the system has zero magnetization, and below a critical temperature it spontaneously picks up an arbitrary direction and develops a constant magnetization. Unlike a ferromagnetic system however, we find a discontinuity in magnetization at the transition temperature. We also study the magnetic susceptibility of various thermodynamic phases of the system and find that depending on temperature and applied magnetic field, a phase is either paramagnetic or diamagnetic.

In the second part, we aim to study transport properties of a parity-odd, non-relativistic charged fluid in presence of background electric and magnetic fields. To obtain the stress tensor and charge current of the non-relativistic system, we start with the most generic relativistic fluid living in one higher dimension, and reduce the constituent equations of the relativistic system along the light-cone direction. This mechanism is known as light-cone reduction. In the similar way, reducing the equation satisfied by the entropy current of the relativistic theory we obtain a consistent entropy current for the non-relativistic system. Demanding the second law of thermodynamics, we impose constraints on various first order transport coefficients (like viscosity, thermal conductivity, electric conductivity etc.) of the fluid. One of our important results is that in $(2+1)$ dimensions, one can have a first derivative, parity-odd fluid only if the fluid is incompressible and is subjected to a constant magnetic field.

List of Notations and Symbols

Part-I: Holographic Ferromagnetism

I	Reissner-Nordström action
G_4	gravitational constant in 4D
$g_{\mu\nu}$	metric
Λ	cosmological constant
b	radius of the AdS spacetime
$\{r, \theta, \phi\}$	polar coordinates
t	time
$\{x^\mu\}$	Minkowski coordinates
r_+	radius of the black hole horizon
M	mass of the blackhole
$R_{\mu\nu}$	Riemann curvature tensor
$F_{\mu\nu}$	EM field tensor
q_E	electric charge
q_M	magnetic charge
ϕ_E	electric potential
ϕ_M	magnetic potential
$T = 1/\beta$	temperature of the black hole
S	entropy of the black hole
W	free energy of the black hole
E	enthalpy of the black hole
$\mathcal{Z}$	partition function
B	magnetic field
$\mathcal{M}$	magnetization
χ	magnetic susceptibility

Part-II: Non-relativistic Charged Hydrodynamics

Relativistic sector

$\{x^\mu\}$	Minkowski coordinates
$\{x^\pm, x^i\}$	light-cone coordinates
d	light-cone spatial dimensions
$g^{\mu\nu}$	metric
u^μ	four-velocity
T	temperature
P	pressure
M_I	chemical potential
E	energy density
Q_I	charge density
R	mass density
S	entropy density
$T^{\mu\nu}$	energy-mom tensor (Eqn. 7.7)
$\Pi^{\mu\nu}$	energy-mom dissipation (Eqn. 7.8)
J_I^μ	charge current (Eqn. 7.7)
Υ_I^μ	charge dissipation (Eqn. 7.8)
θ	velocity gradient (Eqn. 7.9)
$P^{\mu\nu}$	projection operator (Eqn. 7.9)
J_S^μ	entropy current (Eqn. 7.17)
$\tau^{\mu\nu}$	shear viscosity tensor (Eqn. 7.9)
A_I^μ	gauge fields
$F_I^{\mu\nu}$	EM field tensor (Eqn. 7.6)
E_I^μ, B_I^μ	electric/magnetic field (Eqn. 7.6)
η, ζ	shear and bulk viscosities
$\varrho_{IJ}, \lambda_{IJ}, \gamma_I$	charge, electric and thermal conductivities
$\mho_I, \tilde{\mho}_{IJ}$	parity-odd conductivities
$D, \tilde{D}_I$	parity-odd entropy coeff
C_{IJK}	anomaly coefficient
$\theta^{\mu\nu}$	Eqn. (7.9)
$\mathbf{Z}, \mathbf{T}, \mathbf{Y}^{\mu\nu}$	Eqn. (7.14)

Non-relativistic sector	
$\{x^i\}$	Euclidean coordinates
t	time
d	spatial dimensions
g^{ij}, δ^{ij}	Metric
v^i	velocity
τ	temperature
p	pressure
μ_I	chemical potential
ϵ	energy density
q_I	charge density
ρ	mass density
s	entropy density
t^{ij}	stress-energy tensor (Eqn. 7.24)
π^{ij}	stress-energy dissipation (Eqn. 7.25)
j_I^i	charge current (Eqn. 7.24)
ς_I^i	charge dissipation (Eqn. 7.25)
j^i	energy current (Eqn. 7.24)
ς^i	energy dissipation (Eqn. 7.25)
j_S^i	entropy current (Eqn. 8.17)
σ^{ij}	shear viscosity tensor (Eqn. 7.26)
ϕ_I, a_I^i	gauge potentials
$\epsilon_I^i, \beta_I^{ij}$	electric and magnetic fields (Eqn. 7.23)
n, z	shear and bulk viscosity
$\xi_I, \tilde{\xi}_{IJ}, \bar{\xi}_{IJ}$	charge conductivities
$\sigma_I, \tilde{\sigma}_{IJ}, \bar{\sigma}_{IJ}$	electric conductivities
$\kappa, \tilde{\kappa}_I, \bar{\kappa}_I$	thermal conductivities
$\omega_I, \tilde{\omega}_{IJ}$	parity-odd conductivities
$\mathfrak{a}, \mathfrak{b}_I, \mathfrak{c}, \mathfrak{d}_I, \mathfrak{f}_I$	parity-odd entropy coeff
$\alpha_I, \tilde{\alpha}_{IJ}, \bar{\alpha}_{IJ}$	magnetic conductivities
$\tilde{\mu}_I, \bar{\mu}_I$	pressure conductivities
χ	Eqn. (8.13)
ϖ_I	Eqn. (8.14)

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PART - I

Holographic Ferromagnetism

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1 | Introduction and Background

1.1 Holography and the AdS/CFT Conjecture

Holographic Principle is a conjecture in string theory, which predicts that all the information of a (d+1)-dimensional gravitational theory, can be encoded in the (d)-dimensional boundary of the spacetime. The principle was first proposed by Gerard 't Hooft, though Leonard Susskind gave it precise string theoretic interpretation combining ideas from 't Hooft and Charles Thorn.

The idea of holography was inspired from black hole thermodynamics. In the 1970's, Jacob Bekenstein and Stephen Hawking, among many others, realized that black holes have rich thermodynamic structure. Precise expressions for various quantities like entropy, temperature, free energy etc. were given, and were shown to obey the first law of thermodynamics. As it turns out, entropy of a black hole is proportional to the area of its horizon and not its volume, which in some sense is counter intuitive because entropy being an extensive quantity, one would expect it to be proportional to volume. Since entropy of any thermodynamical object is equal to the logarithm of total number of underlying quantum micro-states, it makes us think that all the information (micro-states) of a black hole can be encoded in a theory living in one lower dimension.

Holographic principle by itself is a very strong statement, which till date does not have any precise mathematical derivation or experimental proof. However, string theory provides a canonical example of this holographic principle. The example is known as the AdS/CFT conjecture, proposed by Juan Maldacena in late 1997 [3, 4]. The conjecture relates two seemingly different theories living in different dimensions. The first one is a gravity theory in a $(d+1)$-dimensional asymptotically AdS spacetime (a mathematical spacetime with a constant negative cosmological constant) and the second one is a quantum field theory with conformal invariance[1] living on the d dimensional boundary of the AdS spacetime. The conjecture states that these two theories are dual to each other, *i.e.* when the field theory

[1] Conformal Invariance consists of Lorentz Transformations, spacetime translations, Dilatation and Special Conformal Transformation (SCT).

is in strongly coupled regime the corresponding gravity theory is weakly coupled and vice-versa. Because of this particular character, the conjecture turns out to be an essential tool to deal with strongly coupled quantum field theories. The correspondence has also been applied to study hydrodynamic properties (low energy fluctuations from thermal equilibrium) of different strongly coupled systems. The first attempt was made by Policastro, Son and Starinets in [5, 6]. They calculated the shear viscosity to entropy density ratio of a strongly coupled fluid system and found that it has a universal value. This value was pretty close to the experimentally measured value of shear-viscosity to entropy density ratio of strongly coupled quark-gluon plasma produced at RHIC (Relativistic Heavy Ion Collider) after the collision of two heavy ions.

Recently there has been much interest in holographic study of condensed matter systems. Different properties of $(d+1)$ dimensional condensed matter systems (for example super-conductors, Fermi liquid systems and many more [7, 9, 10]) are being studied from the perspective of string theory.

1.2 Ferromagnetism in a (2+1)-dim Spherical CFT

In the current work, we apply the AdS/CFT conjecture to study magnetic properties of a conformal field theory living on a two-sphere. For that, we consider a $(3+1)$ dimensional dyonic black hole spacetime as our bulk system.

We showed in [1] that when these black holes are considered in an ensemble with constant magnetic charge (q_M) and electric potential at infinity (Φ_E), they exhibit a small-large black hole phase transition. For a particular set of external control thermodynamic parameters, T, Φ_E and q_M there exists multiple solutions of black hole radii (the one with larger radii is termed to be Large Black Hole and one with smaller as Small Black Hole), and the system chooses the one which has lower free energy. The AdS/CFT duality then suggests that the boundary theory, which lives on the surface of a sphere, should also exhibit some kind of phase transition - which we found to be a ferromagnetic-kind phase transition. The boundary theory has zero magnetization at high temperatures, whereas below a transition temperature (analogue of the Curie's point) system instantaneously picks a preferred direction, and has an overall finite but constant magnetization. This should be contrasted with the ferromagnetism we see in usual materials, where below the curie point, magnetization continuously increases with the drop in temperature.

1.3 Summary

We consider a spherical dyonic black hole solution in $(3+1)$-dim asymptotically AdS spacetime. We calculate free energy (W) and temperature (T) of this system. The self intersection of $W - T$ plot (Fig. 1.1) shows that there exist multiple phases in the bulk theory, i.e. at a particular temperature, magnetic charge and electric potential, there exist multiple black hole solutions with different radii and free energies. The bulk theory exhibits a small-large black hole phase transition as the temperature crosses the self intersection point, denoted as the coexistence temperature.

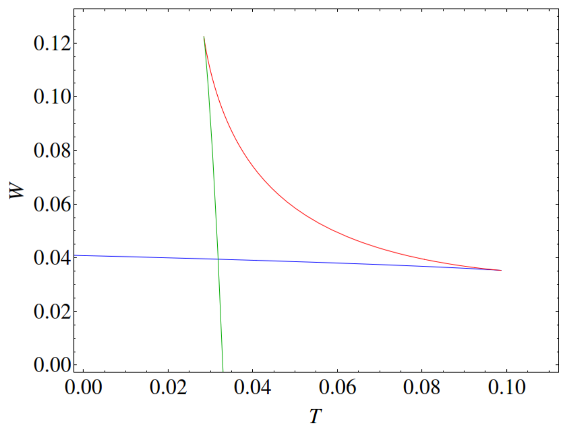


Figure 1.1: $W - T$ plot for all other parameters fixed ($P = 0.0035$, $\Phi_E = 0.8$, $q_M = 0.068$). The graph has three separate branches: Branch-1 (Blue), Branch-2 (Red) and Branch-3 (Green).

We further study the magnetic properties of a $(2+1)$ dimensional boundary CFT, which is dual to our bulk black hole spacetime. The free energy of the CFT is conjectured to be the free energy of the bulk spacetime by the AdS/CFT conjecture. We calculate magnetization of the CFT and find that in $B \to 0$ limit the boundary theory shows a ferromagnetic like behavior. Above a temperature T_o the dominant phase has zero magnetization whereas below T_o a constant magnetization phase is dominant (Fig. 1.2). Unlike a ferromagnetic system though, we find a discontinuity in magnetization at T_0.

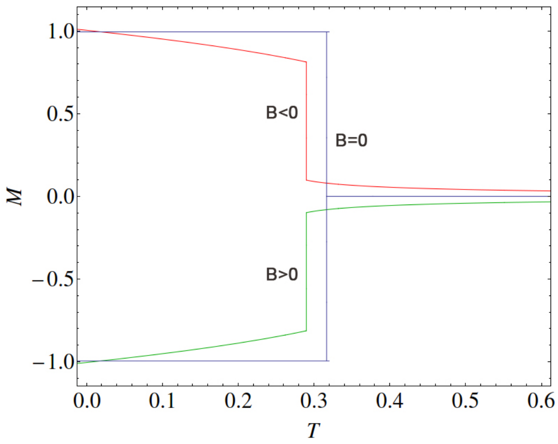
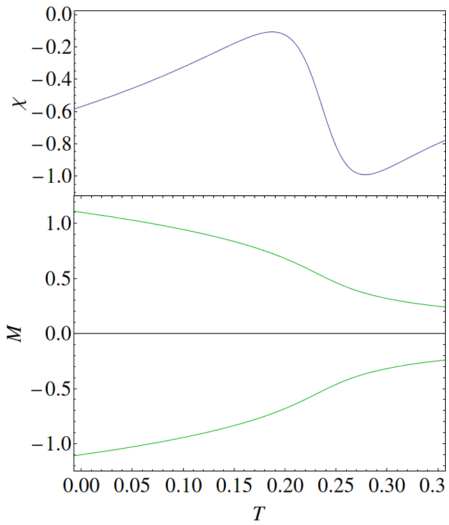


Figure 1.2: $\mathcal{M}$ vs. T curve for $B \to 0$ (blue curve) and $B \neq 0$ (red and green curve for $B < 0$ and $B > 0$ respectively). At the transition temperature a sharp jump in magnetization is observed. Low temperature phase has constant magnetization and high temperature phase has zero magnetization.

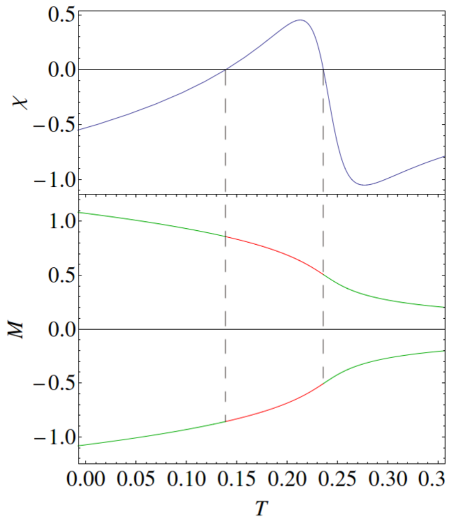
In presence of a finite magnetic field we calculate susceptibility χ of the system and observe diamagnetic ($\chi < 0$) or paramagnetic ($\chi > 0$) behavior depending on the temperature and magnetic field. We brief our observations here and present the details in Section (3):

- There exists a maximum magnetic field B^* above which the thermodynamics is dominated by a diamagnetic phase for any temperature. This phase of CFT is dual to the single black hole phase in the bulk. See Fig. (1.3a).
- For $B_c < B < B^*$ (B_c is the critical magnetic field), the thermodynamics is again governed by a single phase (which is dual to the single black hole phase in bulk). But this phase shows two crossovers between diamagnetic and paramagnetic phases. Very high and very low temperature phases are diamagnetic, while in between the system is paramagnetic. See Fig. (1.3b).
- Criticality appears at $B = B_c$. Two new phases nucleate (one of which is thermodynamically unstable). Other two stable phases are dual to small and large black holes in bulk. If B is sufficiently smaller than B_c, we see a transition between a paramagnetic (dual to SBH) phase and diamagnetic

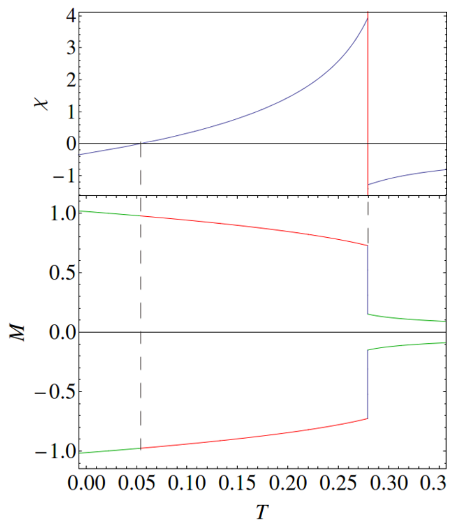
(dual to LBH) phase at $T = T_0$. See Fig. (1.3c).



(a) $B > B^*$.



(b) $B^* > B > B_c$.



(c) B sufficiently less than B_c.

Figure 1.3: $\chi - T$ and $\mathcal{M} - T$ plots. Green segments are diamagnetic while red are paramagnetic. In plot **(c)** the transition from large magnetization phase to small magnetization phase is a transition from a paramagnetic phase to a diamagnetic phase.

2 | Dyonic Black Holes in AdS Spacetime

AdS spacetime is the vacuum solution of Einstein's equations in presence of a negative cosmological constant (Λ), much like Minkowski spacetime is the solution without any Λ. Similar to the Minkowski case [18], we can have a maximally symmetric solution to the Einstein's equations, which will give us a black hole in asymptotically *AdS* space.

We consider the Reissner-Nordström action in four dimensions in presence of a cosmological term, which describes our system of interest:

$$I = \frac{1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\left(-R + F^2 - \frac{6}{b^2}\right). \tag{2.1}$$

Variation of this action gives the equations of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{3}{b^2}g_{\mu\nu} = 2(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}), \qquad \nabla_\mu F^{\mu\nu} = 0, \tag{2.2}$$

maximally symmetric solution to which is given by:

$$A = \left(-\frac{q_E}{r} + \frac{q_E}{r_+}\right)\mathrm{d}t + (q_M\cos\theta)\,\mathrm{d}\phi, \tag{2.3}$$

$$ds^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \tag{2.4}$$

where, A_μ is the $U(1)$ gauge field and

$$f(r) = \left(1 + \frac{r^2}{b^2} - \frac{2M}{r} + \frac{q_E^2 + q_M^2}{r^2}\right). \tag{2.5}$$

q_E, q_M and M are integration constants, identified as electric charge, magnetic charge and mass of the black hole respectively. r_+, the horizon of the black hole is given by the solution of:

$$f(r_+) = \left(1 + \frac{r_+^2}{b^2} - \frac{2M}{r_+} + \frac{q_E^2 + q_M^2}{r_+^2}\right) = 0. \tag{2.6}$$

Subsequently we aim at studying the thermodynamics of the black hole system, for which we choose an ensemble where q_M and the asymptotic value of A_t is constant, which implies that the electric potential Φ_E, defined as:

$$\Phi_E = \frac{q_E}{r_+}, \tag{2.7}$$

is constant in our thermodynamic analysis.

2.1 Hawking Temperature and Free Energy

To study thermodynamic description of a system, one needs the equation of state along with the expression for free energy specific to the ensemble under consideration. In black hole systems, a generic procedure has been prescribed by Gibbons and Hawking in Euclidean framework, to get temperature and free energy.

We define the euclidean time $\tau = \iota t$, which can be shown to be S^1 angular coordinate after appropriate coordinate transformations [19]. In the Euclidean framework we can define the partition function of the system as:

$$\mathcal{Z} = \int [\mathcal{D}g] \mathrm{e}^{-I_E}, \tag{2.8}$$

where I_E is the Euclidean action. In the semi-classical limit that we are considering, the dominant contribution to the path integral comes from classical solutions to the equations of motion. In this case,

$$\log \mathcal{Z} = -I_E^{onshell}, \tag{2.9}$$

where $I_E^{onshell}$ is the action evaluated on equations of motion. Using Eqn. (2.2) we can write

$$I_E^{onshell} = \frac{1}{16\pi G_4} \int \mathrm{d}^4x \sqrt{g} \left(F^2 + \frac{6}{b^2} \right). \tag{2.10}$$

Free energy is thus given by:

$$W = -\frac{1}{\beta} \ln \mathcal{Z} = \frac{I_{onshell}}{\beta}. \tag{2.11}$$

The on-shell action is divergent because the r integration ranges from r_+ to ∞. Therefore, we need to regularize the action by introducing a finite cutoff $\tilde{R}$. Then, to renormalize the action, we can either add some counter-term to the original action or we can subtract the contribution of a background spacetime. Here we follow the first prescription. Method of background subtraction is discussed in Appendix (A.1).

For an electromagnetically charged black hole with mass M, magnetic charge q_M and electric potential at infinity Φ_E, the regularised on-shell action is given by

$$I_{BH} = \frac{1}{16\pi G_4} \int_{r_+}^{\tilde{R}} d^4x \sqrt{g} \left(F^2 + \frac{6}{b^2} \right)$$
$$= \frac{\beta}{4G_4} \left[\frac{2(\Phi_E^2 r_+^2 - q_M^2)}{\tilde{R}} + \frac{6\tilde{R}^3}{3b^2} - \frac{2(\Phi_E^2 r_+^2 - q_M^2)}{r_+} - \frac{6r_+^3}{3b^2} \right]. \tag{2.12}$$

Here $\tilde{R}$ is the cutoff. We shall take $\tilde{R} \to \infty$ at the end. β is the period of Euclidean time coordinate τ, identified as inverse Hawking temperature

$$T = \frac{1}{\beta} = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+} \left[1 + \frac{3r_+^2}{b^2} - \Phi_E^2 - \frac{q_M^2}{r_+^2} \right]. \tag{2.13}$$

This equation serves as the equation of state of the black hole.

Note that the second term in equation (2.12) gives a diverging contribution to free energy as $\tilde{R} \to \infty$. To tame the divergence we add counterterms following [12]. We find that the on-shell action can be made finite with the following counterterms,

$$S_{ct} = \frac{1}{8\pi G_4} \int_{\partial \mathcal{M}} d^3x \sqrt{-\gamma} \left(c_1 + c_2 R^{(3)} \right). \tag{2.14}$$

Here $\partial \mathcal{M}$ is the asymptotic boundary of AdS spacetime, γ is the induced metric on the boundary, $R^{(3)}$ is the Ricci scalar calculated for the metric γ. c_1 and c_2 are two numerical constants, their values are given by

$$c_1 = -\frac{1}{b}, \qquad c_2 = \frac{b}{4}. \tag{2.15}$$

Hence, the free energy is given by,

$$W = \frac{I}{\beta} = \frac{r_+}{4G_4} \left[1 - \frac{r_+^2}{b^2} - \Phi_E^2 + \frac{3q_M^2}{r_+^2} \right]. \tag{2.16}$$

r_+ in the above equation just serves as a parameter, and is to be replaced using Eqn. (2.13) for all purposes. Therefore in our ensemble W is to be understood as $W(T, \Phi_E, q_M)$.

2.2 Black Hole Phase Transition

We now plot $W(T, \Phi_E, q_M)$ with respect to T for everything else held fixed. We find that for a particular temperature, there exists multiple solutions of the black hole, each with different free energy. The solution having least free energy, will dominate at a particular temperature. In figure (1.1), the blue, red and green lines are the free energies for small, unstable and large black hole as a function of temperature. From this plot it is clear that at low temperature there exists only one branch. As we increase temperature two new branches appear (the upper cusp) at $T = T_c$ (black hole nucleation temperature). At this temperature the free energies of two new branches are greater than the first one, hence the first branch dominates the thermodynamics. Upon further increase of temperature, we see that at some temperature ($T = T_o$) the green curve crosses the blue one and after that the free energy of the large black hole dominates over the other two branches. This implies a phase transition between small black hole and large black hole at T_o. There exists a temperature $T_A > T_o$, where Branch-1 and 2 vanish, called the annihilation temperature (T_A).

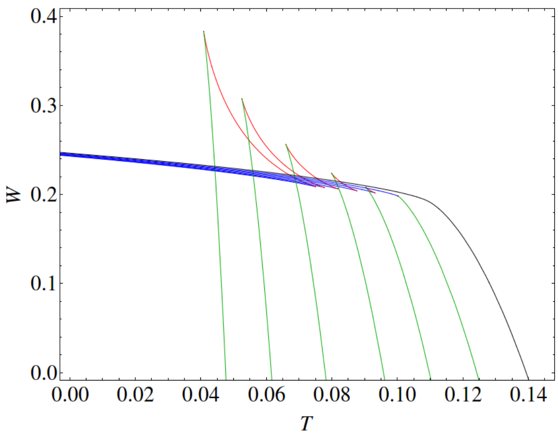


Figure 2.1: $W - T$ diagram for fixed $\Phi_E = 0.5$, $q_M = 0.28$ and varying P. $P \approx 0.018$ is the critical point in this case, after which only one solution of BH remains (the black curve).

The unstable black hole (branch-2, the red curve in the figure 1.1) is always thermodynamically disfavored as the free energy of this branch is greater than the free energy of the other two branches for any temperature. Branch-1 and Branch-3 intersect at the coexistence point T_o (≈ 0.03) which is similar to the liquid-gas phase transition.

The coexistence point can be reached easily by demanding free energy (W) to be same for two different black hole radii: $r_+^{(1)}$ and $r_+^{(2)}$. This is nothing but the

Maxwell's construction. As we vary the temperature, black hole never goes to Branch-2 phase, but jumps from Branch-1 to Branch-3 directly at the coexistence point.

2.2.1 Hawking-Page Phase Transition

In figure (2.2) we plot free energy vs. temperature for different values of q_M, keeping Φ_M fixed. In this section we investigate the $q_M \to 0$ limit carefully. From the plot it is clear that for large q_M we can not see any phase transition (the topmost plot). There exists only one stable black hole. At $q_M = q_{M(c)}$ two new branches appear, one of them is unstable. As we further decrease q_M (> 0) the $W - T$ plot is same as in figure (1.1), as we have already discussed. In $q_M \to 0$ limit however, we see that the blue line (corresponding to small black hole) overlaps with x-axis. This implies that the free energy of the small black hole, in this particular limit, reduces to zero. That is, the small black hole reduces to a global AdS spacetime. Hence, the LBH - SBH phase transition reduces to Hawking-Page phase transition (a transition between black hole and a global AdS spacetime) [13].

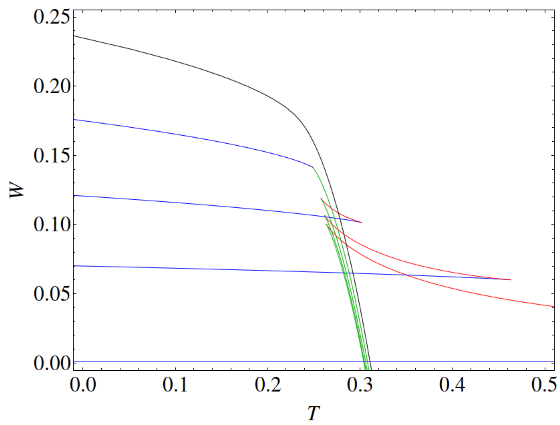


Figure 2.2: $W - T$ for fixed $P = 0.12$, $\Phi_E = 0.045$ and varying q_M. Criticality occurs at $q_M = 0.173$. The bottommost graph is for $q_M = 0$.

An important point to note here is that, in $q_M \to 0$ limit r_+ also goes to zero but the ratio q_M/r_+ remains fixed (which we call Φ_M). Therefore, the global AdS

spacetime has a constant Φ_M. Since we are working in a constant Φ_E ensemble, the global AdS space has a constant electric potential as well.

3 | Thermodynamics of Boundary CFT

(3 + 1)-dim spherical dyonic black hole in asymptotically AdS spacetime is conjectured to be dual to a (2 + 1)-dim CFT living on the boundary of the AdS space, which has topology $R \times S^2$. In this section we will study the implication of small-large black hole phase transition of the bulk on the boundary theory.

3.1 Holographic Dictionary

The bulk gauge field is dual to a global $U(1)$ current operator J_μ. The CFT has a conserved global charge $\langle J^t \rangle$ given by

$$\left\langle J^t \right\rangle \;=\; \frac{q_E}{16\pi G_4} = \frac{\sqrt{2}N^{3/2}q_E}{24\pi b^2} \tag{3.1}$$

where, we use the holographic dictionary

$$\frac{1}{16\pi G_4} = \frac{\sqrt{2}N^{3/2}}{24\pi b^2}, \quad N \text{ is the degree of the gauge group of the } CFT.$$

The boundary CFT also has a constant magnetic field. The strength of the magnetic field is given by $B = q_M/b^2$ which can be read off from the asymptotic value of bulk field strength obtained from equation (2.3). We shall study the magnetic properties of this strongly coupled system using the holographic setup.

In Section (3.2), we see that the CFT undergoes a phase transition. A system in finite volume, in general, does not exhibit any phase transition. But in the large N limit, *i.e.*, when the number of degrees of freedom goes to infinity, it is possible to have a phase transition even in finite volume.

3.2 Magnetization of Boundary Phases

The magnetic field in our boundary theory is given by,

$$B = \frac{q_M}{b^2}. \tag{3.2}$$

When we consider thermodynamics of the boundary theory we define a variable $\mathcal{M}$ (magnetization), conjugate to the external magnetic field B. Different phases of boundary theory are also characterized by this new variable $\mathcal{M}$ defined by the following relation using Eqn. (2.16),

$$\mathcal{M} = -\frac{\partial W}{\partial B}\bigg|_T = -b^2\left(\frac{q_M}{r_+}\right). \tag{3.3}$$

It is worth to note that the positive definiteness of r_+ implies $\mathcal{M}$ and B always have opposite sign.

We calculate free energy and temperature of the CFT in terms of B and $\mathcal{M}$ (boundary parameters) to study the phase structure of the system,

$$W = \frac{1}{4}\left[-(1-\Phi_E^2)b^4\frac{B}{\mathcal{M}} - 3B\mathcal{M} + b^{10}\frac{B^3}{\mathcal{M}^3}\right], \tag{3.4}$$

$$T = \frac{1}{\beta} = \frac{1}{4\pi}\left[-(1-\Phi_E^2)\frac{1}{b^4}\frac{\mathcal{M}}{B} - 3b^2\frac{B}{\mathcal{M}} + \frac{1}{b^8}\frac{\mathcal{M}^3}{B}\right]. \tag{3.5}$$

W in equation (3.4) can be plotted against T for varying B. Since B is proportional to q_M, the plot is the same as in figure (2.2). For B above a critical value B_c (which is proportional to $q_{M(c)}$, discussed in Section 2.2.1) there exists only one phase of boundary CFT and magnetization of this phase is a continuous function of temperature. As B goes below B_c the boundary theory develops two more new phases. Therefore, in this case the theory has three different phases (in a given temperature window). See figure (3.1).

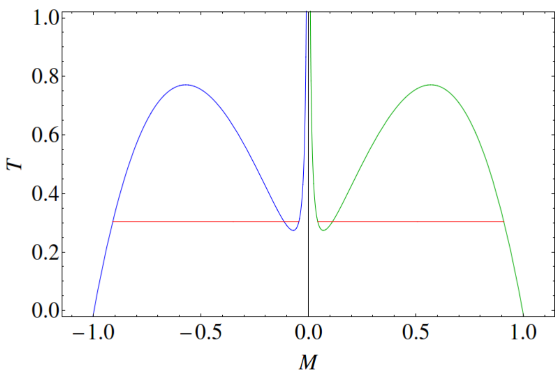


Figure 3.1: $T - \mathcal{M}$ plot and Maxwell's construction. Red line is the co-existence point (corresponds to temperature T_0).

We study the $T - \mathcal{M}$ plots (Fig. 3.1) for our system, which is conventionally done for magnetic systems. The blue (green) curve corresponds to positive (negative) magnetic field. In this figure we see that, below a critical temperature there exists only one phase with high magnetization (dual to small black hole). Above the critical temperature, there are three possible phases with different magnetization. Among them the middle one is thermodynamically unstable. The phase with small magnetization corresponds to the large black hole phase in the dual theory. The red line indicates the transition temperature T_o. Above this temperature, thermodynamics is dominated by the phase with small magnetization. Therefore a sharp jump in magnetization is observed at the transition temperature. In figure (1.2) we plot the same graph removing the unstable branch using Maxwell's construction. $\mathcal{M}$ vs. T plots for different values of magnetic field and B vs. $\mathcal{M}$ plots for different values of temperature are discussed in Appendix (A.2).

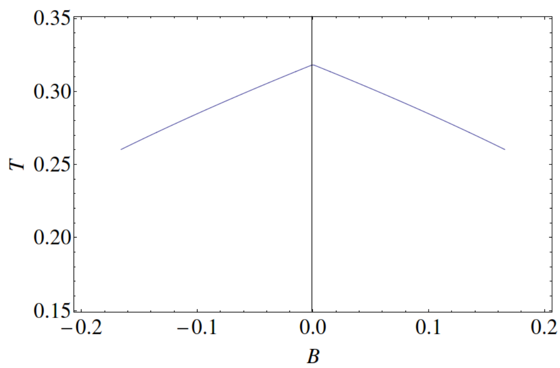


Figure 3.2: T_0 vs. B curve (for $\Phi_E = 0.8$). The end point is the critical point (for a fixed Φ_E). Above (below) the line small (large) magnetization black hole dominates the thermodynamics.

The transition temperature T_o depends on the external magnetic field. As we increase the magnetic field from 0 up to B_c the transition temperature decreases (figure 3.2). For $B > B_c$ there exists only one stable branch.

3.2.1 B $\rightarrow$ 0 limit: Ferromagnetic-like Behavior

$B \rightarrow 0$ limit, in particular is interesting. In this limit we see that the low temperature phase has a constant magnetization whereas the high temperature phase has zero magnetization. There is a transition between zero magnetization phase to constant magnetization phase as we decrease the temperature (See figure 1.2).

Unlike ferromagnetic materials, here we find that the magnetization is discontinuous at the transition temperature. The constant magnetization phase is dual to global AdS phase in the bulk. As we have discussed before, in this limit the radius of small black hole goes to zero with q_M/r_+ fixed. In other words the small black hole evaporates to global AdS with constant Φ_E and $\mathcal{M}$. A more detailed discussion can be found in Appendix (A.3).

As we have explained before, in the limit $B \to 0$, LBH/SBH phase transition reduces to Hawking-Page phase transition. Therefore, low temperature phase (dual to global AdS) of the boundary theory has zero free energy whereas the high temperature phase dual to LBH has free energy of order $N^{3/2}$ in the limit $N \to \infty$. This phase transition is identified with confinement-deconfinement phase transition of gauge theory [14]. Therefore, we see that the confined phase has a constant magnetization whereas the deconfined phase has zero magnetization.

3.3 Magnetic Susceptibility of Boundary Theory

Depending on the temperature and applied magnetic field, boundary theory shows either diamagnetic or paramagnetic behavior. To study the same we calculate magnetic susceptibility using the formula:

$$\chi = \left.\frac{\partial \mathcal{M}}{\partial B}\right|_T . \tag{3.6}$$

A system is said to be diamagnetic if $\chi < 0$ and paramagnetic if $\chi > 0$. The magnetic properties of a physical substance mainly depend on the electrons in the substance. The electrons are either free or bound to atoms. When we apply an external magnetic field these electrons react against that field. In general one can see two important effects. One, the electrons start moving in a quantised orbit in presence of the magnetic field. Two, the spins of the electrons tend to align parallel to the magnetic field. One can neglect the effect of atomic nuclei compared to these two effects, as they are much heavier than electrons. The orbital motion of electrons is responsible for diamagnetism whereas, alignment of electrons' spin along the magnetic field gives rise to paramagnetism. In a physical substance, these two effects compete. In diamagnetic material the first one (orbital motion) is stronger than the second one, and vice-versa in a paramagnetic material.

The temperature in (equation 3.5) can be written in the following differential form for a constant Φ_E:

$$dT = \frac{\partial T}{\partial B} dB + \frac{\partial T}{\partial \mathcal{M}} d\mathcal{M}. \tag{3.7}$$

χ from this expression can be written as

$$\chi = \left.\frac{d\mathcal{M}}{dB}\right|_T = \frac{\partial T}{\partial B}\frac{\partial \mathcal{M}}{\partial T} = \frac{\mathcal{M}}{B}\left(\frac{3b^{10}B^2 + \mathcal{M}^4 - b^4\mathcal{M}^2\left(1-\Phi_E^2\right)}{3b^{10}B^2 + 3\mathcal{M}^4 - b^4\mathcal{M}^2\left(1-\Phi_E^2\right)}\right). \tag{3.8}$$

We plot χ against T for various B to study the magnetic behaviour of the system. Our results are as follows:

1. There exists a magnetic field $B^* > B_c$, above which the CFT is diamagnetic for all temperatures (see figure 1.3a).

2. For $B_c < B < B^*$, still the boundary theory has a single phase but this phase shows two crossovers between paramagnetic and diamagnetic phases. At high and low temperature the system behaves like a diamagnetic system, while in between it shows paramagnetic behavior. See figure (1.3b).

3. For $B < B_c$ (but still close to B_c), the unstable branch of BH pops up. Thus we see a phase transition at T_o from Small BH branch to Large BH branch, both of them being paramagnetic. As temperature is decreased (increased), Small BH (Large BH) branch crosses over to a diamagnetic phase. Figure (3.3) shows the segment of the curves near $T = T_o$.

4. Below B_c, when magnetic field is even below a certain value $B^{\#}$ (discussed in Appendix A.4), the paramagnetic branch of Large BH gets cut off in the Maxwells' construction (figure 1.3c). Thus the phase transition occurs from paramagnetic Small BH to diamagnetic Large BH.

3.3.1 High Temperature Behavior of Magnetic Susceptibility

In the high temperature limit, as we know that only the small magnetization solution (Large BH) dominates, temperature (equation 3.5) has the leading contribution from:

$$T \approx -\frac{3B}{4\pi\mathcal{M}} \tag{3.9}$$

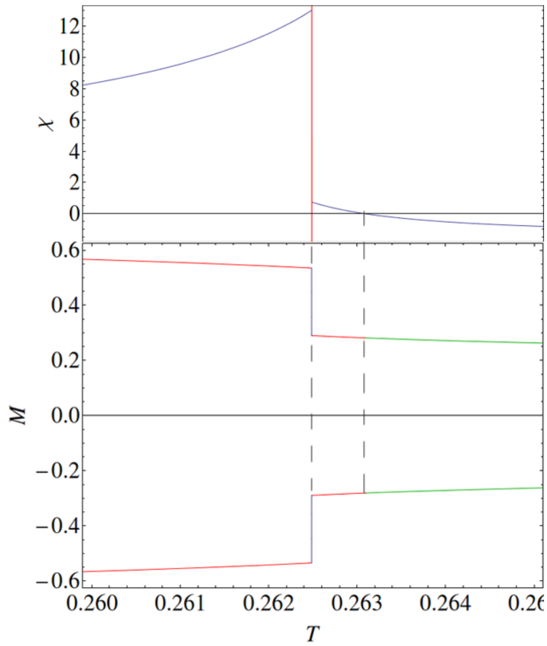


Figure 3.3: $\mathcal{M}$ vs. T and corresponding χ vs. T for $B < B_c$ (but close to it). In this plot it is observed that small magnetization branch has $\chi > 0$ for temperature close to the transition temperature. As we increase temperature, it crosses over to diamagnetic phase $\chi < 0$. Similarly, low temperature phase near the transition temperature hase $\chi > 0$ but as temperature is decreased its susceptibility becomes negative (the plot does not show the low temperature behaviour.)

and χ for large T will thus be given by:

$$\chi \approx \frac{\mathcal{M}}{B} \approx \left(-\frac{3}{4\pi}\right)\frac{1}{T}. \tag{3.10}$$

Hence we see that the Curie's Law is satisfied with a negative Curie Constant $-3/4\pi$.

4 | Stability Analysis

We conclude our discussion by analysing the stability of black hole solutions. When entropy S is a smooth function of extensive variables x_i's then sub-additivity of entropy is equivalent to the Hessian matrix $\left[\frac{\partial^2 S}{\partial x_i \partial x_j}\right]$ being negative definite. For canonical ensemble the only extensive variable is mass (or energy), therefore, sub-additivity of entropy implies that $C_P > 0$. For grand canonical ensemble, the variables are mass and charges therefore the stability lines are determined by finding the zeroes of the determinant of the Hessian matrix. It has been argued in [15] that the zeroes of the determinant of the Hessian of S with respect to M and q_i's coincide with the zeroes of the determinant of the Hessian of the Gibbs (Euclidean) action,

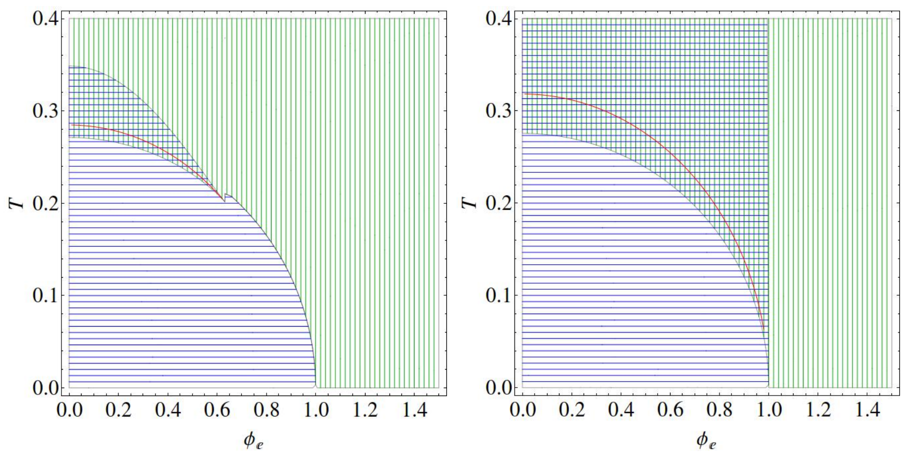
$$I_G = \beta \left(M - \sum_i q_i^{\text{phys}} \Phi_i \right) - S \tag{4.1}$$

with respect to r_+ and q_i's keeping β and Φ_i's fixed. Note that q_i's are the charge parameters entering into the black hole solutions where q_i^{phys}'s are the physical charges. Though this criteria can figure out the instability line in the phase diagram but it is unable to tell which sides of the phase transition lines correspond to local stability. One can figure out the stability region by using the fact that zero chemical potential and high temperature must correspond to a stable black hole solution.

We compute the zeroes of the Hessian of free energy $W = M - \Phi_E q_E - TS$ with respect to q_E and r_+ keeping T and Φ_E fixed, which gives us one condition (or bound) on the phase space. Positivity of temperature will give another condition. These two conditions give the following bound on the phase space.

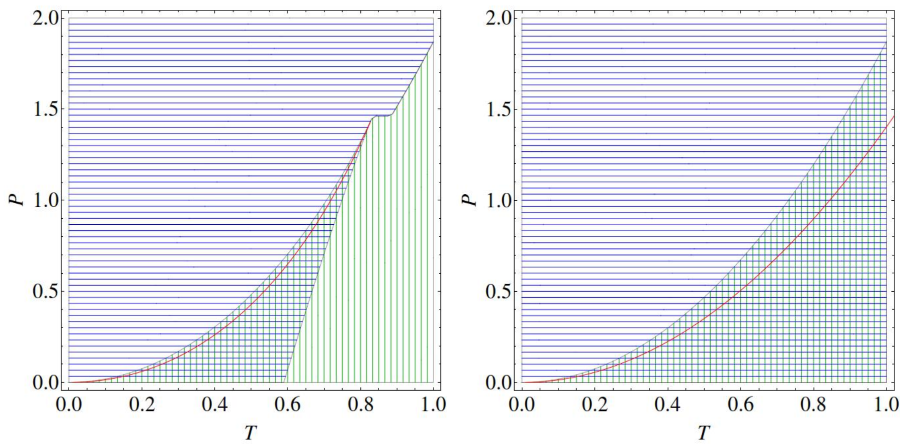
$$3q_M^2 + \frac{3r_+^4}{b^2} - r_+^2(1 - \Phi_E^2) > 0, \qquad -q_M^2 + \frac{3r_+^4}{b^2} + r_+^2(1 - \Phi_E^2) > 0. \tag{4.2}$$

In figure (4.1) and (4.2) We plot these stability lines and find that all values of q_M, Φ_E and $T > 0$ results in a stable solution.



(a) (b)

Figure 4.1: Stability plots in $T - \Phi_E$ phase space for **(a)** $q_M < q_{M(c)}$ and **(b)** $q_M = 0$. Horizontal blue lines correspond to Small BH solution stability region, whereas vertical green lines correspond to the stability region of Large BH. Red line corresponds to the phase transition.



(a) (b)

Figure 4.2: Stability plots in $P - T$ phase space for **(a)** $q_M < q_{M(c)}$ and **(b)** $q_M = 0$. Horizontal blue lines correspond to Small BH solution stability region, whereas vertical green lines correspond to the stability region of Large BH. Red line corresponds to the phase transition.

5 | Discussion

In this work we studied thermodynamic properties of a $(2+1)$ dimensional fluid system living on the boundary of a sphere. We found that system exhibits a ferromagnetic-kind phase transition, which is quite similar qualitatively to the usual ferromagnetic phase transitions. Only difference is that the magnetization is no longer a continuous function of temperature, and we see a sudden jump of magnetization from 0 to a finite value about the critical point.

The ferromagnetic-kind phase transition in the boundary theory is a manifestation of the Hawking-Page phase transition in the bulk. The background AdS of bulk corresponds to finite magnetization phase at the boundary, and the large black hole corresponds to zero magnetization phase. As is pointed out in [14], the Hawking-Page phase transition corresponds to confinement-deconfinement phase transition in the gauge theories. We hence confer that confined phase has constant magnetization, while deconfined phase has zero magnetization.

In [16, 17] the authors discussed different phases of boundary CFT which are dual to different bulk solutions. They considered a unitary matrix model in the weak coupling side and showed that in the large N limit, there exists three different saddle points which correspond to small, large and unstable black holes in the bulk. The key point of their observation is that in the canonical ensemble, the fixed electric charge constraint contributes an additional logarithmic term $\log(\mathrm{Tr}U\mathrm{Tr}U^{\dagger})$ involving the order parameter, to the gauge theory effective action. However, in our case, we have a constant magnetic field in the boundary theory. It is a good exercise to understand the effect of this constant magnetic field in the effective action of boundary theory.

PART - II

Non-relativistic Charged Hydrodynamics

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6 | Introduction and Background

6.1 Hydrodynamics

Hydrodynamics is an effective description of nearly equilibrium interacting many body systems. Fluid systems are considered to be continuous, i.e. when we talk about an infinitesimal volume element (or 'fluid particle'), it still contains a large number of molecules. More specifically, the size of a fluid particle is much much greater than the mean free path of the system. In the same spirit, velocity of the fluid at a point is to be considered as the velocity of the respective fluid particle, and not the velocities of molecules themselves.

A fluid is completely determined by its velocity $\vec{v}(\vec{x},t)$ and the set of independent thermodynamic parameters (like pressure $p(\vec{x},t)$, energy density $\epsilon(\vec{x},t)$, temperature $\tau(\vec{x},t)$, mass density $\rho(\vec{x},t)$, charge density $q(\vec{x},t)$ etc., out of which not all are independent due to thermodynamic relations and equation of state) as a function of space and time. The flow of a fluid is governed by a set of equations known as constitutive equations, which are essentially the conservation equations for mass, energy, momentum and charge.

The equations of hydrodynamics assume that the fluid is in local thermodynamic equilibrium at each point in space and time, even though different thermodynamic quantities may vary. Therefore fluid mechanics essentially applies only when the length scales of variation of thermodynamic variables are large compared to equilibration length scale of the fluid, namely mean free path [20].

Since fluids are not in 'absolute' thermodynamic equilibrium, second law of thermodynamics tells us that there should not be any local loss of entropy. The increase of entropy at every spacetime point is called internal friction, viscosity or dissipation. The dissipation arises due to gradient of thermodynamic quantities in the fluid which takes it away from the equilibrium, and hence is accounted for in the constitutive equations by means of derivative dependence of mass/energy/momentum/charge flow on fluid variables. However, since the fluid is assumed to be in local equilibrium, the derivatives are fairly small, and one can perform a derivative expansion of the theory, to study it order by order in derivatives. The zeroeth order fluid is called the 'ideal fluid', i.e. a fluid without any dissipation. We consider in this work a fluid in first order derivative expansion. The coefficients coupling to various derivative terms in constitutive equations are called transport coefficients - they describe the strength of effect of fluctuations of a thermodynamic parameter on the respective equations. These transport coefficients can in general be a function of the fluid's thermodynamic parameters but not velocity.

As it turns out, not all derivative terms produce entropy; instead some end up reducing it. Therefore one must explicitly look at the entropy current, to find out what derivative terms are allowed in the constitutive equations and under what conditions. Equivalently, entropy current positivity imposes certain constraints on the transport coefficients of the fluid, while certain transport coefficients are turned off. The same has been established for uncharged fluids in Landau's book [20]. One of the goals of this work is to write a consistent entropy current for charged non-relativistic fluids and get constraints on various transport coefficients.

Landau, in his book on fluid dynamics [20] gives a thorough discussion on uncharged, non-relativistic as well as relativistic fluids upto first derivative order. In subsequent years, the charged fluids has been also fairly curated and understood. In Section (7) we will review the key features of relativistic and non-relativistic fluid dynamics.

A relativistic fluid theory can obviously be reduced to a non-relativistic theory under the special case of $v \ll c$ or $c \to \infty$, under certain assumptions[1]. For charged fluids this limit has been well studied in [26], and we review its basic layout in Appendix (B.2). It is shown in [27] however, that one can also reach to a $(d+1)$-dim non-relativistic fluid theory starting from $(d+2)$-dim relativistic theory by following a mechanism known as Light Cone Reduction (LCR). We shall discuss more about Light Cone Reduction in the next section. One of the aims of this work is to construct a consistent non-relativistic fluid description using LCR, and compare it to the description already obtained by non-relativistic limit in [26].

[1] The assumption involved is that the two limits - non-relativistic limit and continuity limit of fluids, commute, which as of yet has no underlying reasons to be true.

6.2 Light Cone Reduction

Discrete Light Cone Quantization is a well established mechanism in Quantum Field Theories, which connects a $(d+2)$-dim relativistic theory to a $(d+1)$-dim non-relativistic theory. The mechanism involves a coordinate transformation in relativistic theory from the Minkowski coordinates $\{x^\mu\}_{\mu=0,1,..,d+1}$ to the light cone coordinates $\{x^\pm, x^i\}_{i=1,2..,d}$:

$$x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^{d+1}\right). \tag{6.1}$$

We now let the theory evolve in x^+ direction and identify it with the new 'time' coordinate (t) of the non-relativistic theory, keeping the x^- coordinate fixed (i.e. ∂_- acting on any parameter of the theory will vanish). Under this identification the new theory in $(d+1)$-dim with coordinates $\{t, x^i\}_{i=1,2..,d}$ respects the non-relativistic symmetry [24]. The reduction essentially means that the underlying symmetry group of relativistic theory (eg. Poincaré or Conformal) reduces to the underlying symmetry group of the non-relativistic theory (eg. Galilean or Schrödinger respectively).

In some sense, this way of looking at a non-relativistic theories is much more proper, because the 'Galilean Invariance' is fundamental backbone of a non-relativistic system. On the other hand however, the small velocity limit of a relativistic theory is bound to give a non-relativistic theory, but there is no unique and consistent way to do it. Especially in fluids, we have an underlying caveat that the two limits: hydrodynamic and non-relativistic, might not commute.

In the current work we perform LCR on a $(d+2)$-dim relativistic hydrodynamic theory, and obtain a $(d+1)$-dim non-relativistic theory. It has been already shown in [25] that this gives a consistent non-relativistic fluid in uncharged case, except that the extensivity is lost during reduction. Here we extend this idea to charged non-relativistic fluids as well. We discuss more about this issue in Section (8). Authors in [26] already studied such fluids via a $1/c$ expansion; we will discuss disparities in their and our results and reasons for them to arise.

6.3 Summary

We are interested in the properties of a charged non-relativistic fluid. As was discussed in Section (6.2), a consistent way to get a non-relativistic fluid theory is to perform light cone reduction of a one dimensional higher relativistic fluid theory. Under the LCR identification, the constitutive equations of the relativistic theory reduce to the constitutive equations of the non-relativistic theory, and one can find a one to one mapping between all the parameters of either theories.

One can always put some allowed parity odd dissipative terms in the constitutive equations of the fluid. However it can be checked that at one derivative order the parity odd terms can only be kept in at most (4)-dimensional relativistic theories. For theories in more that four dimensions, the parity-odd terms only appear at higher derivative orders. Here we will restrict ourselves to relativistic theories in at least four dimensions, as theories in further lower dimensions have non-trivial parity-odd dependence. We especially treat the $(3+1)$-dim relativistic fluid (with parity-odd terms), which upon reduction gives a $(2+1)$-dim non-relativistic fluid. We are interested in studying the effects of the parity odd terms on such fluids.

To include the both, $(d+2)$-dim relativistic theories for $d=2$ (where parity-odd terms appear) and $d \geq 2$, in one setting we introduce a 'parity odd notation': all the parity odd terms will be denoted in a pair of braces $\{\cdots\}$ with an understanding that the braces will only contribute for $d=2$.

We start with a generic, first derivative order, charged relativistic fluid theory in $(d+2)$-dim in the presence of some background electromagnetic fields. After LCR we find the non-relativistic fluid variables and transport coefficients, in terms of the respective relativistic quantities. We consistently obtain the stress-energy tensor (t^{ij}), energy current (j^i) and charge current (j^i_I) of the non-relativistic fluids:

$$t^{ij} = v^i v^j \rho + p g^{ij} - n\sigma^{ij} - z u^+ g^{ij} \nabla_k v^k, \tag{6.2}$$

$$j^i = v^i \left(\epsilon + p + \frac{1}{2}\rho \mathbf{v}^2\right) - n\sigma^{ik} v_k - z v^i \nabla_k v^k - \kappa \nabla^i \tau - \tau \sigma_I \nabla^i \left(\frac{\mu_I}{\tau}\right)$$
$$+ \sigma_I(\epsilon^i_I - v_j \beta^{ji}_I), \tag{6.3}$$

$$j_I^i = q_I v^i - \tilde{\kappa}_I \nabla^i \tau - \tilde{\xi}_{IJ} \nabla^i \left(\frac{\mu_J}{\tau}\right) - \tilde{m}_I \nabla^i p + \tilde{\sigma}_{IJ} \left(\epsilon_J^i - v_k \beta_J^{ki}\right)$$
$$+ \left\{ \bar{\kappa}_I \epsilon^{ij} \nabla_j \tau + \bar{\xi}_{IJ} \epsilon^{ij} \nabla_j \left(\frac{\mu_J}{\tau}\right) - \bar{m}_I \epsilon^{ij} \nabla_j p + \bar{\sigma}_{IJ} \epsilon^{ij} \left(\epsilon_{Jj} - v^k \beta_{Jkj}\right) \right\}. \quad (6.4)$$

Explanation and definitions of the involved quantities are provided in the main text.

In charge current (Eqn. 6.4) we identify the thermal Hall contribution $\bar{\kappa}_I \epsilon^{ij} \nabla_j \tau$ (where τ is the temperature of non-relativistic fluid). The Hall energy flow sourced by temperature gradient is known as the Leduc-Righi effect. It has been predicted in condensed matter physics, that this kind of effect can be observed in various topological insulators. We also obtain an anomalous electromagnetic Hall energy current $\bar{\sigma}_{IJ} \epsilon^{ij} \left(\epsilon_{Jj} - v^k \beta_{Jkj}\right)$ where, ϵ_{Jj} is electric field and β_{Iij} is proportional to the magnetic field. We have also verified the famous "Wiedemann-Franz Law" for metals which predicts the ratio $\tilde{\sigma}_{IJ}/\kappa$ to be inversely proportional to τ, that was not reproduced in [26].

Finally we obtain the entropy current j_S^i for the non-relativistic charged fluids by LCR, and argue its positivity to determine constraints on various transport coefficients. The constraints however internally consistent, are quite different from the $1/c$ expansion [26], especially for the transport coefficients of charge current. The disparity can be accounted to a fundamental difference between the two theories: $1/c$ expansion of a relativistic theory with extensivity gives an extensive non-relativistic theory, however LCR breaks down the extensivity upon reduction. Moreover, the fluid upon LCR does not necessarily have a constant charge to mass ratio. If we impose this constraint to the LCR reduced system, we find that all the transport coefficients, barring the bulk viscosity z, vanish.

Another interesting result we gain is that, entropy positivity allows parity odd terms only for an incompressible non-relativistic fluid in $(2+1)$-dim, if it is kept in an electromagnetic field with constant magnetic field. Otherwise all the parity-odd terms must vanish. Remember that for our system, we have no parity-odd terms in higher dimensions at first derivative order.

7 | Review of Hydrodynamics

Let us start our discussion with a quick review of the vital points in relativistic and non-relativistic hydrodynamics. A charged fluid with only one kind of constituent particle in $(d, 1)$ dimensions (d spatial and 1 time dimensions) can be completely determined by $d+3$ fluid variables[1]: fluid velocity (u^μ/v^i) (only d velocity components of u^μ are independent due to normalization $u^\mu u_\mu = -1$) and intensive thermodynamic parameters - temperature (T/τ), pressure (P/p) and chemical potential (M_I/μ_I) (I index corresponds to the multiple $U(1)$ charges introduced in the fluid, which is important in various branches of physics. One can just fix $I = 1$ to reach the conventional single charge case). Extensive thermodynamic parameters - energy density (E/ϵ), entropy density (S/s) and charge density (Q_I/q_I) are not considered to be independent as they are determined by the first law of thermodynamics[2]:

$$\mathrm{d}E = T\mathrm{d}S + M_I \mathrm{d}Q_I + (E + P - ST - Q_I M_I)\frac{\mathrm{d}R}{R}, \tag{7.1}$$

and the equation of state. Here, (R/ρ) is the mass density of the fluid in its local rest frame. It is not really an independent parameter as mass to charge ratio of particles is constant. Additionally we have an Euler's relation which follows from the extensivity of internal energy:

$$E + P = TS + Q_I M_I, \tag{7.2}$$

which in conjugation with the first law, allows us to drop one more thermodynamic parameter. In which case we can specify the fluid system with $d+2$ independent parameters, chosen to be: T, M_I/T and u^μ.

However as we shall see later, Light Cone Reduction does not preserve extensivity and thus we do not have Eqn. (7.2) in a non-relativistic theory obtained from LCR. Hence we are forced to consider all the $d+3$ variables independent: p, τ, μ_I/τ and v^i.

[1]In our notation among the pairs like (E/ϵ), first one are relativistic quantities and the second one are non-relativistic.

[2]The thermodynamic laws here are mentioned in terms of densities, which can be trivially derived from original laws under the assumption that total mass of the system is constant. Same laws are also valid for non-relativistic variables.

7.1 Relativistic Hydrodynamics

We consider a relativistic charged fluid in $(d+1,1)$ dimensions. The $d+3$ independent fluid parameters (T, M_I/T and u^μ) are related by $d+3$ equations of motion of the fluid known as constitutive equations. They are nothing but the conservation equations of energy-momentum and charge:

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu J^\mu_I = 0, \tag{7.3}$$

where $T^{\mu\nu}$ is the energy-momentum tensor and J^μ_I is the charge current. Therefore once $T^{\mu\nu}$ and J^μ_I are known, fluid is completely determined (along with the equation of state of-course). For an ideal relativistic fluid (in absence of viscosity or internal friction) they are given as:

$$T^{\mu\nu} = (E+P)u^\mu u^\nu + P g^{\mu\nu}, \qquad J^\mu = Q_I u^\mu. \tag{7.4}$$

7.1.1 Viscous Relativistic Fluids in Presence of Background Fields

An important characteristic of fluids is dissipation. Since fluid is a macroscopic system, it is governed by the second law of thermodynamics, which says that total entropy of a system should always increase. Since fluid is assumed to be in thermodynamic equilibrium at every space-time point, entropy should be created locally at every point. We will look at it explicitly later, but for now it suffices to mention that for a system in exact thermodynamic equilibrium (ideal fluid), the entropy is constant. However, if we slowly depart from the equilibrium by introducing gradients of control parameters $(T, M_I/T, u^\mu)$ all over the fluid, the fluid starts to create entropy. This phenomenon is called dissipation.

Since fluid dynamics is nearly in equilibrium, we suppose that the gradients of $T, M_I/T, u^\mu$ are fairly small, and hence one can rest upon a perturbative expansion in derivatives. In this work we consider a fluid upto first derivative order.

Further, the fluid can be introduced to some external electromagnetic gauge fields A^μ, which affect its dynamics. The energy-momentum and charge will no longer be conserved, and hence we have:

$$\nabla_\mu T^{\mu\nu} = F_I^{\nu\alpha} J_{I\alpha}, \qquad \nabla_\mu J_I^\mu = \{C_{IJK} E_J^\mu B_{K\mu}\}, \tag{7.5}$$

where we have the electromagnetic field tensor:

$$F_I^{\mu\nu} = \nabla^\mu A_I^\nu - \nabla^\nu A_I^\mu, \qquad E_I^\mu = F_I^{\mu\nu} u_\nu, \qquad B_I^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{I\alpha\beta}. \tag{7.6}$$

C_{IJK} is called the anomaly coefficient, which is a characteristic of 4-dimensional fluid theories. Refer the $\{\cdots\}$ convention in Section (6.3).

In presence of electromagnetic fields and viscosity, the energy-momentum and charge currents are modified to:

$$T^{\mu\nu} = (E+P)u^\mu u^\nu + P g^{\mu\nu} + \Pi^{\mu\nu}, \qquad J_I^\mu = Q_I u^\mu + \Upsilon_I^\mu, \tag{7.7}$$

where we have the dissipative terms:

$$\begin{aligned} \Pi^{\mu\nu} &= -2\eta\tau^{\mu\nu} - \zeta\theta^{\mu\nu}, \\ \Upsilon_I^\mu &= -\varrho_{IJ} P^{\mu\nu} \nabla_\nu \left(\frac{M_J}{T}\right) + \lambda_{IJ} E_J^\mu - \gamma_I P^{\mu\nu} \nabla_\nu T + \left\{ \mho_I l^\mu + \tilde{\mho}_{IJ} B_J^\mu \right\}, \end{aligned} \tag{7.8}$$

$$\begin{aligned} l^\mu &= \epsilon^{\mu\alpha\beta\gamma} u_\alpha \nabla_\beta u_\gamma, \\ \tau^{\mu\nu} &= \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} \left[\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d+1} g_{\alpha\beta}\theta \right], \\ \theta^{\mu\nu} &= \theta P^{\mu\nu}, \qquad \theta = \nabla_\alpha u^\alpha, \qquad P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \end{aligned} \tag{7.9}$$

whereas velocity normalization is given by:

$$u^\mu u_\mu = -1. \tag{7.10}$$

η and ζ are relativistic shear and bulk viscosity coefficients respectively. ϱ_{IJ}, λ_{IJ} and γ_I are relativistic charge, electric and thermal conductivities. Finally $\mho_I$ and $\tilde{\mho}_{IJ}$ are some relativistic parity-odd transport coefficients. We will see later however that not all of these transport coefficients are independent.

It should be noted that we have included all the possible derivative terms in the dissipation in the chosen basis $(T, M_I/T, u^\mu)$, which are consistent with the Landau gauge[3]:

$$u_\mu \Pi^{\mu\nu} = 0, \qquad u_\mu \Upsilon_I^\mu = 0, \tag{7.11}$$

[3] Landau gauge assumes that there is no dissipation in the direction of fluid flow [20].

Now we have a complete description of relativistic charged fluids in electromagnetic background upto one derivative order. But, since we are not interested in higher derivative orders, we can use Eqn. (7.5) at the first derivative level:

$$u^\alpha \nabla_\alpha E = -(E+P)\theta, \qquad P^{\mu\alpha}\nabla_\alpha P - Q_I E_I^\mu = -(E+P)u^\alpha \nabla_\alpha u^\mu,$$
$$u^\mu \nabla_\mu Q_I + Q_I \theta = \{C_{IJK} E_J^\mu B_{K\mu}\}. \tag{7.12}$$

By doing this we are introducing an error at second derivative order, which is to be neglected. We have also used $E_I^\mu u_\mu = 0$, which can be checked trivially from Eqn. (7.6). Using these $\tau^{\mu\nu}$ from Eqn. (7.9) can be written as:

$$\tau^{\mu\nu} = \frac{1}{2}\left[\mathbf{Y}^{\mu\nu} + \mathbf{Y}^{\nu\mu} - \frac{2}{d+1} g^{\mu\nu}\theta - \mathbf{Z} u^\mu u^\nu\right], \tag{7.13}$$

where,

$$\mathbf{Z} = \frac{2}{d+1}\frac{u^\alpha \nabla_\alpha \mathbf{T}}{(E+P)}, \qquad \mathbf{T} = (d+1)P - E,$$
$$\mathbf{Y}^{\mu\nu} = \nabla^\mu u^\nu - \frac{u^\nu \nabla^\mu P}{E+P} + Q_I \frac{u^\nu E_I^\mu}{E+P}. \tag{7.14}$$

$\mathbf{T}$ has been defined for a convenient switchover to conformal fluids, which are of abundant interest in physics; for instance in the first part of this thesis report. Conformal fluids are those whose energy-momentum tensor is traceless:

$$T^\mu_{\ \mu} = \mathbf{T} - \zeta(d+1)\theta = 0, \tag{7.15}$$

which can be reached just by setting $\mathbf{T} = \zeta = 0$.

7.1.2 Entropy Current

Now we turn our attention to the second law of thermodynamics:

$$\nabla_\mu J_S^\mu \geq 0. \tag{7.16}$$

Considering the possibility of creating entropy at every space-time point, we added the most generic viscous terms to the constitutive equations of the fluid. But it turns out that not all those terms always create entropy, instead some end up reducing it. Or in other words, entropy current positivity cannot be ensured in the presence of certain dissipative terms with arbitrary transport coefficients. Hence we get some constraints on these transport coefficients.

The canonical entropy current is given by:

$$J_S^\mu = Su^\mu - \frac{M_I}{T}\Upsilon_I^\mu, \tag{7.17}$$

Using Eqn. (7.12) one can then show that:

$$\nabla_\mu J_S^\mu = -\frac{1}{T}\Pi^{\mu\nu}\nabla_\mu u_\nu + \left[\frac{E_{I\mu}}{T} - \nabla_\mu\left(\frac{M_I}{T}\right)\right]\Upsilon_I^\mu, \tag{7.18}$$

which only depends on the dissipative terms. This implies that in absence of dissipation fluid does not create any entropy. Plugging in values of $\Pi^{\mu\nu}$ and Υ^μ from Eqn. (7.8) and demanding $\nabla_\mu J_S^\mu \geq 0$, we will obtain the constraints:

$$\begin{gathered}\gamma_I = 0, \quad \eta \geq 0, \quad \zeta \geq 0, \quad \lambda_{IJ} = \frac{1}{T}\varrho_{IJ}, \\ \varrho_{IJ} \quad \text{matrix is positive definite.}\end{gathered} \tag{7.19}$$

However there are additional parity-odd terms in the charge current that cannot be made positive definite, corresponding to coefficients $\mho_I$ and $\tilde{\mho}_{IJ}$. One would then expect that these coefficients must vanish. But a fabulous idea was proposed in [21], where authors modify the entropy current itself with the most generic parity odd vectors:

$$J_S^\mu \to J_S^\mu + \left\{Dl^\mu + \tilde{D}_I B_I^\mu\right\}, \tag{7.20}$$

demanding that the contributions of all parity odd terms collectively must vanish. This demand relates $\mho_I, \tilde{\mho}_{IJ}, D, \tilde{D}_I$ and C_{IJK} as follows:

$$\begin{gathered}\frac{\partial D}{\partial P} = \frac{2D}{E+P}, \qquad \frac{\partial D}{\partial(M_I/T)} = \mho_I, \qquad \frac{\partial \tilde{D}_J}{\partial P} = \frac{\tilde{D}_J}{E+P}, \qquad \frac{\partial \tilde{D}_J}{\partial(M_I/T)} = \tilde{\mho}_{IJ}, \\ D\frac{2Q_I}{E+P} - 2\tilde{D}_I = -\frac{1}{T}\mho_I, \qquad \tilde{D}_J\frac{Q_I}{E+P} = -\frac{1}{T}\tilde{\mho}_{IJ} + C_{KIJ}\frac{M_K}{T}.\end{gathered} \tag{7.21}$$

Now we have a complete description of generic relativistic fluids, with all the first order dissipative term allowed by entropy positivity.

7.2 Non-relativistic Fluid Dynamics

Let us now consider a first derivative order, non-relativistic charged fluid in $(d,1)$ dimensions, in presence of electromagnetic fields. It would be helpful to contrast the theory with its relativistic analogue to gain some understanding of similarities and distinctions. The $d+3$ independent fluid parameters[4] (p, τ, μ_I/τ and v^i) are related by $d+3$ constitutive equations, which are the conservation equations of mass, energy, momentum and charge, respectively:

$$\begin{aligned}\partial_t\rho+\partial_i(\rho v^i)&=0, \qquad \partial_t\left(\epsilon+\frac{1}{2}\rho\mathbf{v}^2\right)+\partial_i j^i=j^i_I\epsilon_{Ii},\\ \partial_t q_I+\partial_i j^i_I&=0, \qquad \partial_t(\rho v^j)+\partial_i t^{ij}=q_I\epsilon^j_I-j_{Ii}\beta^{ij}_I,\end{aligned} \tag{7.22}$$

where we can introduce the scalar potential (ϕ) and vector potential (a^i), in terms of which electric and magnetic fields are given as:

$$\epsilon^i_I=-\partial^i\phi_I-\partial_t a^i_I, \qquad \beta^{ij}_I=\partial^i a^j_I-\partial^j a^i_I. \tag{7.23}$$

$-j_{Ii}\beta^{ij}_I$ translates to $q_I(\vec{v}\times(\vec{\nabla}\times\vec{a}_I))^i$ in four dimensional non-viscous fluid. Stress-energy tensor, energy current and charge current are respectively given by:

$$t^{ij}=\rho v^i v^j+pg^{ij}+\pi^{ij}, \qquad j^i=\left(\epsilon+p+\frac{1}{2}\rho\mathbf{v}^2\right)v^i+\varsigma^i, \qquad j^i_I=q_I v^i+\varsigma^i_I, \tag{7.24}$$

where we have the dissipative terms:

$$\begin{aligned}\pi^{ij}&=-n\sigma^{ij}-z\delta^{ij}\partial_k v^k\\ \varsigma^i&=-n\sigma^{ij}v^j-z\partial_k v^k v^i-\kappa\partial^i\tau-\xi\nabla^i\left(\frac{\mu_I}{\tau}\right)+\sigma_I\epsilon^i_I,-\alpha_I v_j\beta^{ji}_I,\\ \varsigma^i_I&=-\tilde{\kappa}_I\nabla^i\tau-\tilde{\xi}_{IJ}\nabla^i\left(\frac{\mu_J}{\tau}\right)-\tilde{m}_I\nabla^i p+\tilde{\sigma}_{IJ}\epsilon^i_J-\tilde{\alpha}_{IJ}v_k\beta^{ki}_J\\ &\quad+\left\{\bar{\kappa}_I\epsilon^{ij}\nabla_j\tau+\bar{\xi}_{IJ}\epsilon^{ij}\nabla_j\left(\frac{\mu_J}{\tau}\right)-\bar{m}_I\epsilon^{ij}\nabla_j p+\bar{\sigma}_{IJ}\epsilon^{ij}\epsilon_{Jj}-\bar{\alpha}_{IJ}\epsilon^{ij}v^k\beta_{Jkj}\right\}.\end{aligned} \tag{7.25}$$

and σ^{ij} is defined as:

$$\sigma^{ij}=\partial^i v^j+\partial^j v^i-\delta^{ij}\frac{2}{d}\partial_k v^k. \tag{7.26}$$

Here n and z are non-relativistic bulk and shear viscosity coefficients respectively. κ, ξ_I, σ_I and α_I are thermal, charge, electric and magnetic conductivities. $\tilde{\kappa}_I,\tilde{\sigma}_{IJ},\tilde{\xi}_{IJ},\tilde{m}_I,\tilde{\alpha}_{IJ}$ and $\bar{\kappa}_I,\bar{\sigma}_{IJ},\bar{\xi}_{IJ},\bar{m}_I,\bar{\alpha}_{IJ}$ are some corresponding arbitrary

[4] Recall that we are not assuming Euler's relation to hold in the non-relativistic theory.

parity-even and parity-odd transport coefficients[5]. Later we shall see that after light cone reduction of a relativistic fluid theory most of these coefficients are related.

Note that we are considering mass and charge continuity equations separately, because as we shall see later, light cone reduction does not enforce mass to charge ratio of particles to be constant (which is not very encouraging).

This finishes our discussion on the fluid mechanical machinery we would be requiring for further analysis. Interested readers can find a detailed discussion on fluid dynamics in [20, 23].

[5]The sign of these coefficients are completely arbitrary for now, and are chosen keeping in mind later convenience.

8 | Light Cone Reduction

Now that we have reviewed the necessary aspects of hydrodynamics, we are ready to continue with the main subject matter of this work. As was discussed in Section (6.2), LCR is a mechanism which connects a $(d+1,1)$-dim relativistic fluid to a $(d,1)$-dim non-relativistic fluid. Essentially we perform the following coordinate transformation to the relativistic theory in Minkowski coordinates $\{x^\mu\}_{\mu=0,1,..,d+1}$:

$$\sqrt{2}x^\pm = x^0 \pm x^{d+1}, \qquad ds^2 = -2dx^+dx^- + \sum_{i=1}^{d}(dx^i)^2, \tag{8.1}$$

to go to light-cone coordinates $\{x^\pm, x^i\}_{i=1,2,..,d}$. Then, we reduce the theory along x^- direction, identifying x^+ with the non-relativistic time (t). This reduces the underlying symmetry group of relativistic theory (e.g. Poincaré or Conformal) to the symmetry group of the non-relativistic theory (e.g. Galilean or Schrödinger respectively).

Light cone reduction of the constitutive equations of a relativistic fluid boil down to the non-relativistic constitutive equations in one dimension lower. Relativistic charged fluid in $(d+1)+1$ dimensions, as we have already discussed, has $d+3$ independent variables: $T, M_I/T, u^\mu$. On reduction, the non-relativistic fluid in d spatial dimensions also has total $d+3$ independent variables: $p, \tau, \mu_I/\tau, v^i$. This is because upon reduction system loses its extensivity, which we will check explicitly later. Our goal is to find a mapping between these independent variables.

8.1 Reduction of Background Fields

Let us start with reducing the background electromagnetic fields. Maxwell's equations for the relativistic system are given by:

$$\nabla_\mu F_I^{\mu\nu} = 0. \tag{8.2}$$

We take the fields to be sourceless, as we assume that our fluid is very weakly charged to effect the background field configuration. Under light-cone reduction the above equations take the following form:

$$\vec{\nabla}^2 A_I^+ = 0, \quad \nabla_i \left(\nabla^i A_I^- + \nabla_+ A_I^i\right) = -\nabla_+^2 A_I^+, \quad \nabla_i \left(\nabla^i A_I^j - \nabla^j A_I^i\right) = \nabla^j \nabla_+ A_I^+. \tag{8.3}$$

These equations can be identified with source free static Maxwell's equations of a non-relativistic system if we map[1]:

$$A_I^- = \phi_I, \qquad A_I^i = a_I^i, \qquad A_I^+ = \text{constant} \tag{8.4}$$

8.2 Reduction of Energy-Momentum and Charge

The reduction of relativistic equations of energy-momentum and charge conservation after using Eqn. (8.4) gives:

$$\begin{aligned}
\nabla_+ T^{++} + \nabla_i T^{i+} &= 0, \\
\nabla_+ T^{+-} + \nabla_i T^{i-} &= -J_I^i \left(\nabla_i A_I^- + \nabla_+ A_{Ii}\right), \\
\nabla_+ T^{+j} + \nabla_i T^{ij} &= -J_I^+ \left(\nabla_+ A_I^j + \nabla^j A_I^-\right) - J_I^i \left(\nabla_i A_I^j - \nabla^j A_{Ii}\right), \\
\nabla_+ J_I^+ + \nabla_i J_I^i &= 0,
\end{aligned} \tag{8.5}$$

which reduce to non-relativistic equations under following identifications:

$$T^{++} = \rho, \qquad T^{i+} = \rho v^i, \qquad T^{+-} = \epsilon + \frac{1}{2}\rho \mathbf{v}^2, \qquad T^{i-} = j^i, \qquad T^{ij} = t^{ij},$$
$$J_I^+ = q_I, \qquad J_I^i = j_I^i. \tag{8.6}$$

Carrying out this identification at first derivative order will explicitly give the following form of non-relativistic fluid variables:

$$\rho = (E+P)(u^+)^2 + (u^+)^2 \left(\eta \mathbf{Z} - \zeta\theta\right), \qquad v^i = \frac{u^i}{u^+} - \frac{\eta}{\rho}\mathbf{Y}^i,$$
$$p = P - \left(\frac{\eta}{d}\mathbf{Z} - \zeta \frac{u^\alpha \nabla_\alpha P}{E+P}\right), \qquad \epsilon = \frac{1}{2}(E-P) - \frac{1}{2}\left(\eta \mathbf{Z} - \zeta\theta\right),$$
$$\tau = \frac{T}{u^+} + \mathcal{O}(1), \qquad \mu_I = \frac{M_I}{u^+} + \mathcal{O}(1),$$
$$j^i = v^i \left(\epsilon + p + \frac{1}{2}\rho \mathbf{v}^2\right) - n\sigma^{ik} v_k - z v^i \nabla_k v^k - \kappa \nabla^i \tau - \tau \sigma_I \nabla^i \left(\frac{\mu_I}{\tau}\right) + \sigma_I(\epsilon_I^i - v_j \beta_I^{ji})$$

[1] In Appendix (B.1) we have discussed direct non-relativistic limit of Maxwell's equations.

$$q_I = u^+ Q_I - u^+ \left[\varrho_{IJ} u^\nu \nabla_\nu \left(\frac{M_J}{T} \right) + \gamma_I u^\nu \nabla_\nu T + \left\{ \mho_I \epsilon^{ij} u^+ \nabla_i v_j + \tilde{\mho}_{IJ} \epsilon_{ij} \beta^{ij}_J \right\} \right]$$

$$\begin{aligned} j^i_I &= q_i v^i - \tilde{\kappa}_I \nabla^i \tau - \tilde{\xi}_{IJ} \nabla^i \left(\frac{\mu_J}{\tau} \right) - \tilde{m}_I \nabla^i p + \tilde{\sigma}_{IJ} \left(\epsilon^i_J - v_k \beta^{ki}_J \right) \\ &+ \left\{ \bar{\kappa}_I \epsilon^{ij} \nabla_j \tau + \bar{\xi}_{IJ} \epsilon^{ij} \nabla_j \left(\frac{\mu_J}{\tau} \right) - \bar{m}_I \epsilon^{ij} \nabla_j p + \bar{\sigma}_{IJ} \epsilon^{ij} \left(\epsilon_{Jj} - v^k \beta_{Jkj} \right) \right\}. \end{aligned} \tag{8.7}$$

and transport coefficients:

$$\begin{gathered} n = \eta u^+, \qquad z = \zeta u^+, \qquad \kappa = 2n \frac{\epsilon + p}{\tau \rho}, \qquad \xi = \tau \sigma_I = n \frac{q_I \tau}{\rho} \\ \tilde{m}_I = \frac{\gamma_I \tau u^+}{2(\epsilon + p)}, \qquad \tilde{\xi}_{IJ} = \left[\varrho_{IJ} + \frac{\xi_J q_I}{2(\epsilon + p)} - m_I q_J \tau \right], \\ \tilde{\kappa}_I = \frac{\kappa q_I}{2(\epsilon + p)}, \qquad \tilde{\sigma}_{IJ} = \tilde{\alpha}_{IJ} = \left[\lambda_{IJ} u^+ + \frac{\sigma_J q_I}{2(\epsilon + p)} \right], \\ \bar{m}_I = \frac{2\omega_I}{\rho}, \qquad \bar{\xi}_{IJ} = \xi_J \frac{\omega_I}{n}, \qquad \bar{\kappa}_I = \kappa \frac{\omega_I}{n}, \qquad \bar{\sigma}_{IJ} = \bar{\alpha}_{IJ} = \left[\sigma_J \frac{\omega_I}{n} - 2\tilde{\omega}_{IJ} \right], \end{gathered} \tag{8.8}$$

where,

$$\omega_I = \mho_I (u^+)^2, \qquad \tilde{\omega}_{IJ} = \tilde{\mho}_{IJ} u^+. \tag{8.9}$$

Note that after identification not all transport coefficients are independent; only independent coefficients are: $n, z, \tilde{m}_I$, $\tilde{\xi}_{IJ}$, $\tilde{\sigma}_{IJ}$, ω_I and $\tilde{\omega}_{IJ}$.

Wiedemann-Franz Law: This famous law predicts the ratio of charge conductivity (which appears in charge current)[2] to thermal conductivity (which appears in energy current) in metals as: $\tilde{\sigma}/\kappa = 1/L\tau$, where L is the Lorenz number predicted to be $\sim 2.45 \times 10^{-8}\ W\Omega K^{-2}$. The law is found to be in good agreement with experiments. We attempt to check the same in our setup[3]:

$$\frac{\tilde{\sigma}}{\kappa} = \frac{\rho \varrho}{2n(\epsilon + p)} + \frac{\tau q^2}{4(\epsilon + p)^2}, \tag{8.10}$$

We model the electrons in metals as free classical gas with no external pressure: fluid with homogeneous particles each of charge e (electronic charge), mass m_e

[2]There is only one $U(1)$ charge here.

[3]We have used here the fact that non-relativistic system respects the constraint $\tau \tilde{\sigma}_{IJ} = \tilde{\xi}_{IJ}$, which has been showed in next section.

and average energy $3/2k_B\tau$. One can check that under mentioned assumptions, our system follows Wiedemann-Franz Law with Lorenz number given as:

$$L = \left(\frac{\varrho}{n}\frac{m_e}{3k_B} + \frac{e^2}{9k_B^2}\right)^{-1}, \tag{8.11}$$

Assuming ϱ and n of nearly same order, first term turns out to be about 15 orders of magnitude smaller than the second term and can be safely neglected. Hence Lorenz number is our case is given approximately by: $6.68 \times 10^{-8}\ W\Omega K^{-2}$, given our assumptions, which is in fair agreement with the experimental value.

8.3 Reduction of Thermodynamics

If we claim that the parameters of the non-relativistic theory are indeed thermodynamic parameters, they must satisfy the first law of thermodynamics (Eqn. 7.1) at all derivative orders. Demanding it will tell us the identification:

$$s = Su^+(1+\chi), \tag{8.12}$$

where χ is determined by the solution of:

$$TSu^\mu\nabla_\mu\chi + M_I u^\mu\nabla_\mu\varpi_I + M_I\varpi_I\theta = -\eta\frac{d+1}{2d}\mathbf{Z}^2 - \zeta\theta^2 + \zeta(u^+)^2\left(\nabla_k v^k\right)^2, \tag{8.13}$$

$$\varpi_I = -\left[\varrho_{IJ}u^\nu\nabla_\nu\left(\frac{M_J}{T}\right) + \gamma_I u^\nu\nabla_\nu T + \left\{\mho_I\epsilon^{ij}u^+\nabla_i v_j + \tilde{\mho}_{IJ}\epsilon_{ij}\beta_J^{ij}\right\}\right]. \tag{8.14}$$

However, after this identification one can check that the Euler's equation (Eqn. 7.2) is not valid anymore, and instead we have:

$$2(\epsilon + p) = s\tau + q_I\mu_I. \tag{8.15}$$

This implies that after light cone reduction, the non-relativistic theory is no longer extensive; or in other words LCR breaks extensivity. The possibility of same has been described in [27].

8.4 Reduction of Entropy Current

We have now a complete non-relativistic theory, with all the parameters and transport coefficients determined in terms of the relativistic ones. The relativistic transport coefficients, as we know, are constrained by entropy current positivity in relativistic theory, which also follows to the non-relativistic transport coefficients due to LCR identification. However, the actual constraints on the non-rel transport coefficients should be derived by asking entropy current positivity independently in the non-relativistic theory; and there is no reason a priori to assume entropy positivity in either theories to be equivalent. In fact we will establish that the equivalence holds only in the parity-even sector, while in the parity-odd sector it is broken.

Reduction of the relativistic entropy current in Eqn. (7.18), using all the previous identifications will give:

$$\nabla_+ s + \nabla_i j^i_S = \frac{1}{\tau}\frac{n}{2}\sigma^{ij}\sigma_{ij} + \frac{1}{\tau}z(\nabla_k v^k)^2 + \frac{1}{\tau}\nabla^i\left[\kappa\nabla_i\tau + \tau\sigma_I\nabla^i\left(\frac{\mu_I}{\tau}\right) - \sigma_I(\epsilon^i_I - v_k\beta^{ki}_I)\right] + \left[\frac{\epsilon_{Ji} - v^k\beta_{Jki}}{\tau} - \nabla_i\left(\frac{\mu_J}{\tau}\right)\right]\varsigma^i_I, \quad (8.16)$$

where we have defined the canonical entropy current as:

$$j^i_S = sv^i - \frac{\mu_I}{\tau}\varsigma^i_I, \quad (8.17)$$

which looks quite natural if one compares it to the chargeless case in [20]. Plugging in the expression for ς^i_I (the dissipative part of non-relativistic charge current) from Eqn. (8.7), and demanding $\nabla_+ s + \nabla_i j^i_S \geq 0$, we will get the constraints:

$$\tilde{m}_I = 0, \quad n \geq 0, \quad z \geq 0, \quad \tilde{\sigma}_{IJ} = \frac{1}{\tau}\tilde{\xi}_{IJ},$$
$$\left[\tilde{\xi}_{IJ} - \frac{\xi_J q_I}{2(\epsilon + p)}\right] \quad \text{matrix is positive definite,} \quad (8.18)$$

in parity even sector, which are exactly what implied by the identifications Eqn. (8.8) and relativistic constraints Eqn. (7.19). However similar to the relativistic case, we find some parity-odd terms in the entropy current which does not guarantee entropy positivity. Thus we need to add the most generic parity odd terms to the entropy current, and hope to get a consistent set of differential equations, allowing us to retain the parity odd terms in the non-relativistic description. Once that is done, we will have a complete description of our non-relativistic fluid.

8.4.1 Parity-odd corrections to NR Entropy Current

For non-relativistic fluid theories in more that two spatial dimensions, we have a complete and consistent description, as the troublesome parity-odd terms in Eqn. (8.16) do not even appear. However, for fluids in two spatial dimensions, we must add some parity-odd terms to the entropy current, if we wish to preserve parity-odd terms in our theory. We make the most generic parity-odd modification to the entropy current in two spatial dimensions as follows:

$$j_S^i \to j_S^i + \left\{ \mathfrak{a}\epsilon^{ij}\nabla_j\tau + \mathfrak{b}_I\epsilon^{ij}\nabla_j\left(\frac{\mu_I}{\tau}\right) + \mathfrak{c}\epsilon^{ij}\nabla_j p + \mathfrak{d}_I\epsilon^{ij}\epsilon_{Ij} + \mathfrak{f}_I\epsilon^{ij}v^k\beta_{Ikj} \right\}, \quad (8.19)$$

such that all the parity odd terms cancel each other to zero. Demanding so, one finds the following set of constraints must be followed:

$$\frac{\partial\mathfrak{a}}{\partial p} - \frac{\partial\mathfrak{c}}{\partial\tau} = 0, \qquad \frac{\partial\mathfrak{a}}{\partial(\mu_I/\tau)} - \frac{\partial\mathfrak{b}_I}{\partial\tau} = \frac{\epsilon+p}{\tau}\frac{2\omega_I}{\rho}, \qquad \frac{\partial\mathfrak{b}_I}{\partial p} - \frac{\partial\mathfrak{c}}{\partial(\mu_I/\tau)} = \frac{2\omega_I}{\rho}, \quad (8.20)$$

$$-\frac{\partial\mathfrak{f}_I}{\partial\tau} = \frac{\partial\mathfrak{d}_I}{\partial\tau} = \frac{\epsilon+p}{\tau^2}\frac{2\omega_I}{\rho}, \qquad \frac{\partial\mathfrak{f}_I}{\partial p} = -\frac{\partial\mathfrak{d}_I}{\partial p} = \frac{2\omega_I}{\tau\rho},$$

$$-\frac{\partial\mathfrak{f}_I}{\partial(\mu_J/\tau)} = \frac{\partial\mathfrak{d}_I}{\partial(\mu_J/\tau)} = \frac{1}{\rho}\left(\omega_I q_J + \omega_J q_I - 2\rho\tilde{\omega}_{JI}\right), \quad (8.21)$$

and additionally, the following two matrices must be demanded to be symmetric:

$$\left[2\rho\tilde{\omega}_{IJ} - \omega_I q_J\right], \qquad \left[\frac{\rho}{t}\frac{\partial\mathfrak{b}_J}{\partial(\mu_I/t)} - \omega_I q_J\right], \quad (8.22)$$

which will make all but the following two terms vanish:

$$\left\{\mathfrak{f}_I\epsilon^{ij}\beta_{Iij}\nabla_k v^k - \mathfrak{d}_I\epsilon^{ij}\nabla_+\beta_{Iij}\right\} \quad (8.23)$$

It shall be noted however that the new coefficients introduced in j_S^i need not be most generic, and their only significance is to make the residual parity-odd terms in entropy current vanish. Hence any special and simple solution to the above constraints is acceptable. We make a special choice as follows:

$$\mathfrak{a} = \mathfrak{c} = 0, \qquad \mathfrak{f}_I = -\mathfrak{d}_I, \quad (8.24)$$

which solves the respective constraints, and leaves us with only two independent coefficients in entropy current: $\mathfrak{b}_I$ and $\mathfrak{d}_I$.

Now we turn attention towards the last two remaining terms in entropy current (Eqn. 8.23). They will vanish only under following two scenarios:

1. $\mathfrak{d}_I \neq 0$: incompressible flow ($\nabla_k v^k = 0$) in electromagnetic background with constant magnetic field ($\nabla_+ \beta_{Iij} = 0$).

2. Setting $\mathfrak{d}_I = 0$ otherwise.

2nd scenario is rather trivial, as it would imply that ω_I, $\tilde{\omega}_{IJ}$ and hence $\mathfrak{b}_I$, all must vanish. Consequently all parity-odd terms will be eradicated from entire theory, making the case similar to the fluids in more that 2 spatial dimensions.

It should be noted that physicality forces us to ask entropy positivity in non-relativistic regime, where we want to study our system, and not the relativistic regime from where we started. Relativistic system can just be considered as a mathematical model, and might not be a 'physical' system. However at least for parity-even sector it turns out that demanding positivity entropy in relativistic or non-relativistic regime is equivalent, as they give same constraints Eqn. (7.19) and (8.18). But for parity-odd terms relativistic entropy positivity and non-relativistic entropy positivity are two different set of constraints. For example, C_{IJK} do not even appear in non-relativistic relations Eqn. (8.20) - (8.22), however relativistic relations Eqn. (7.21) relate $\mho_I$ and $\tilde{\mho}_{IJ}$ to C_{IJK}. Therefore even if we choose to start with a 'physical' relativistic fluid, we are forced to constraint the parameters further for the reduced non-relativistic theory to make sense. However for making the non-relativistic theory 'physical', the beginning relativistic theory need not be demanded 'physical'.

The constraints $\omega_I = \tilde{\omega}_{IJ} = 0$ in 2nd scenario are consistent with the NR entropy current positivity, though are inconsistent with the entropy current positivity in relativistic sector (Eqn. 7.21). Therefore the relativistic fluid we are starting with is 'unphysical', though it does not matter since the non-relativistic system of interest is physical.

The first scenario will be treated in greater detail in next section.

8.5 Incompressible Fluid in Constant Magnetic Field

We have seen that only case where the parity-odd terms can survive, is an incompressible fluid kept in a magnetic field with constant magnetic component in

(2+1)-dim. Equation of state of an incompressible fluid is given by:

$$\rho = \text{constant}, \tag{8.25}$$

which implies that $\nabla_k v^k = 0$, or in other words, there is no compression or expansion of the fluid during the flow. Using the non-relativistic constitutive equations one can easily show that:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + (\epsilon + p)\nabla_k v^k = v_j \nabla_i \pi^{ij} - \nabla_i \varsigma^i, \tag{8.26}$$

which means that for incompressible ideal fluids energy is conserved, and starts to dissipate only in presence of viscosity.

In a (2+1)-dim system with background fields, non-relativistic Maxwell's equations (see Appendix B.1) are given by:

$$\nabla_i \epsilon_I^i = 0, \qquad \nabla_i \beta_I^{ij} = 0, \qquad 2\epsilon^{ij}\nabla_i \epsilon_{Ij} + \epsilon^{ij}\nabla_+ \beta_{Iij} = 0. \tag{8.27}$$

Second equation implies that magnetic field is a pseudo-scalar field constant in space. Also if magnetic field β_I^{12} is time independent (as we demand), electric field is curl free. But first equation already tells us that electric field should be divergence-free, so electric field is constant over the 2D space. However, electric field can still be time dependent, as the corresponding term does not appear in the magnetic limit of maxwell's equations. So finally in the current case we have, a constant over space but time-varying electric vector field and a space-time constant magnetic pseudo-scalar field.

We have checked that the form of j_S^i needed to preserve parity-odd terms is:

$$j_S^i = s v^i - \frac{\mu_I}{\tau} v_I^i + \mathfrak{b}_I \epsilon^{ij} \nabla_j \left(\frac{\mu_I}{\tau}\right) + \mathfrak{d}_I \epsilon^{ij} \left(\epsilon_{Ij} - v^k \beta_{Ikj}\right), \tag{8.28}$$

where $\mathfrak{b}_I$ and $\mathfrak{d}_I$ are determined by equations (8.20) - (8.22). Hence we will get the positivity of entropy along with constraints Eqn. (8.18). However ω_I and $\tilde{\omega}_{IJ}$ remain unconstrained by entropy positivity.

9 | Discussion

We started with a generic $(d+2)$-dim non-conformal relativistic fluid ($d \geq 2$) with anomalies specific to $d = 2$ in presence of background fields on flat space. Light cone reduction of this system gave a $(d+1)$-dim non-relativistic fluid with anomalies (specific to $d = 2$). On physical basis of entropy positivity, we deduced that parity odd terms in non-relativistic theory can only sustain for incompressible fluid in electromagnetic background with constant magnetic field in $(2+1)$-dim. Otherwise for a generic fluid in $(d+1)$-dim ($d > 2$), or for compressible fluid, or a fluid in time-variable magnetic field in $(2+1)$-dim, the parity odd terms must disappear.

The theory we gain by LCR is purely 'Galilean', as the reduction is done at the level of algebra. Therefore we have generic[1] dissipative terms in our non-relativistic fluid, which are allowed by the symmetry and the entropy positivity. LCR does not constraint the size of these coefficients however, and they as well might be infinitesimal and hence not present in practical conditions. On the contrary, when one performs a $1/c$ expansion [22], many of these terms are suppressed depending on the physical considerations and the type of system under view. We discuss the basic aspects of $1/c$ expansion in Appendix (B.2).

Apart from the dissipation terms, ours and the non-relativistic system of [22] have certain fundamental differences. Firstly, their system is 'extensive' as the thermodynamic variables follow Euler's relation; however since LCR breaks the Euler's relation, our system is no longer extensive. Secondly, in our system ρ is not necessarily proportional to q_I, while in [22] it is true at least at the leading $1/c$ order. This is why our non-relativistic system has one more independent parameter as opposed to the $1/c$ case. We can however enforce $\rho \propto q_I$ in our system as well, but it turns out that demanding so switches off all the dissipative terms from the theory except for the bulk viscosity.

In [22] authors do not present a entropy current calculation for non-relativistic fluid obtained by $1/c$ expansion. In fact, as we will review in Appendix (B.2), the

[1]Word generic here just mean possible, and not all possible. We might have more or less terms, depending on the type of relativistic theory we are starting with and the identification.

entropy positivity in turns out to be trivial, and is just followed from the leading order entropy current of the relativistic theory. The constraints on the transport coefficients (which survives at the leading order) also turns out to be the same. However in our case, we have slightly different constraints, because of the above mentioned fundamental differences between the two cases.

PART - III

Appendix

A | Holographic Ferromagnetism

A.1 Background Subtraction

We calculate on-shell action by subtracting the contribution of an extremal magnetically charged black hole with magnetic charge q_M. The reason to choose this particular background depends on our choice of ensemble (see [11]). The extremal black solution is given by equation (2.3) and (2.4) with $q_E \to 0$

$$f_e(r) = \left(1 + \frac{r^2}{b^2} - \frac{2M_e}{r} + \frac{q_M^2}{r^2}\right). \tag{A.1}$$

We consider that the extremal black hole has a constant electric potential which is same as Φ_E of the black hole. The black hole has a horizon radius r_e, given by

$$f_e(r_e) = \left(1 + \frac{r_e^2}{b^2} - \frac{2M_e}{r_e} + \frac{q_M^2}{r_e^2}\right) = 0. \tag{A.2}$$

The extremality condition is given by

$$r_e^2 + \frac{3r_e^4}{b^2} = q_M^2. \tag{A.3}$$

Therefore, the on-shell action of background turns out to be

$$I_E = \frac{1}{16\pi}\int_{r_e}^{R} d^4x\sqrt{g}\left(F^2 + \frac{6}{b^2}\right) = \frac{\beta'}{4}\left[\frac{-2q_M^2}{R} + \frac{6R^3}{3b^2} - \frac{-2q_M 2}{r_e} - \frac{6r_e^3}{3b^2}\right] \tag{A.4}$$

where, β' is the radius of the Euclidean time circle of background. β' can be obtained by identifying the asymptotic boundary geometry of the black hole spacetime and extremal black hole spacetime

$$\beta\sqrt{g_{\tau\tau}} = \beta'\sqrt{g^e_{\tau\tau}}.$$

Thus, we get

$$\beta' = \beta\left(1 - \frac{(M - M_e)b^2}{R^3}\right). \tag{A.5}$$

Subtracting the contribution of extremal black hole on-shell action from that of black hole we finally get,

$$I_{onshell} = I_{BH} - I_E = \frac{\beta}{4}\left[-\Phi_E^2 r_+ + \frac{3q_M^2}{r_+} - \frac{r_+^3}{b^2} + r_+ - 4r_e - \frac{8r_e^3}{b^2}\right]. \tag{A.6}$$

Hence, the free energy is given by,

$$W = \frac{I}{\beta} = \frac{r_+}{4}\left[-\Phi_E^2 + \frac{3q_M^2}{r_+^2} - \frac{r_+^2}{b^2} + 1 - 4r_e - \frac{8r_e^3}{b^2}\right]. \tag{A.7}$$

This renormalization prescription tells us that all the thermodynamic quantities we measure (for example energy, volume, charges etc.) are with respect to an extremal magnetic black hole background.

A.2 Critical Points and $\mathcal{M}$ vs. B Plots

W in equation (3.4) can be plotted against T for varying B. Since B is identified with q_M, the plot is the same as figure (2.2). This time Branch-1 corresponds to high magnetization (Small black hole), and Branch-3 corresponds to low magnetization (Large black hole). We note the key points of Small-Large BH phase transition in this language:

Critical Curve is given by:

$$T_c = \frac{2}{\pi\sqrt{6}b}(1 - \Phi_{E(c)}^2)^{1/2} \tag{A.8}$$

or correspondingly:

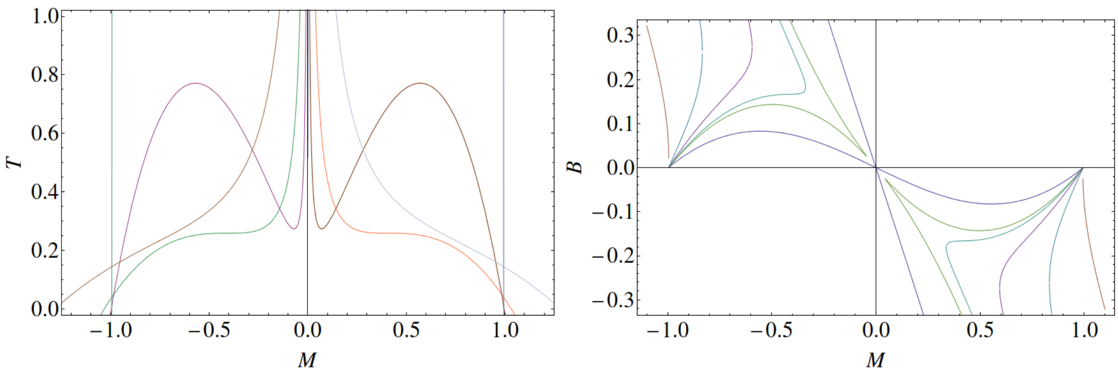
$$B_c = \pm\frac{1}{6b}(1 - \Phi_{E(c)}^2) \tag{A.9}$$

while, Magnetization $\mathcal{M}$ at critical point(s) is given by:

$$\mathcal{M}_c = \mp\frac{b^2}{\sqrt{6}}(1 - \Phi_{E(c)}^2)^{1/2}. \tag{A.10}$$

A thing to note here will be the Nucleation temperature, T_N. As $B \to 0$, T_N saturates to some maximum value $T_{N(M)}$. Same is evident in figure (A.1b), (A.1a) ($T_{N(M)} \approx 0.3$). It is the temperature when a curve (In figure A.1a) touches the origin. At $T > T_{N(M)}$ (representative blue graph), $B = 0$ has 3 solutions. $T_{N(M)}$ will be given by the minima of $T - \mathcal{M}$ graph as $B \to 0$, which using equation 3.5 is:

$$T_{N(M)} = \frac{\sqrt{3}}{2\pi b}\sqrt{1 - \Phi_E^2}. \tag{A.11}$$



(a) (b)

Figure A.1: **(a)** $T - \mathcal{M}$ graph for varying B ($\Phi_E = 0.1$). In order of the graphs near the T axis for small values of $\mathcal{M}$, the leftmost is for the highest positive B, and the rightmost for highest negative (magnitude) B. Criticality occurs in Green ($\mathcal{M} < 0$) and Light Orange ($\mathcal{M} > 0$) graphs ($B = \pm 0.165$). **(b)** $B - \mathcal{M}$ graph for varying T ($\Phi_E = 0.1$). In order of the graphs in the second quadrant, the leftmost is for the lowest T and the rightmost for highest T. Curves in the fourth quadrant are the segments of the same graphs. Criticality occurs at $T = 0.258$.

Similar to the nucleation temperature, coexistence temperature T_o also reaches its maximum value $T_{o(M)}$ as $B \to 0$. It can be easily found by finding root of W which gives:

$$T_{o(M)} = \frac{1}{\pi b}\sqrt{1 - \Phi_E^2}. \tag{A.12}$$

The coexistence point for the phase transition can also be translated similarly demanding W to be same for two phases, which will give corrected $B - \mathcal{M}$ and $T - \mathcal{M}$ plots. Coexistence curve (and plane) is however found to be like figure (A.2).

A.3 Magnetization from the Minima of Free Energy

As we have seen that our system possesses a 'ferromagnetic' type behaviour in $B \to 0$ limit. In this appendix we investigate possible minima of the free energy

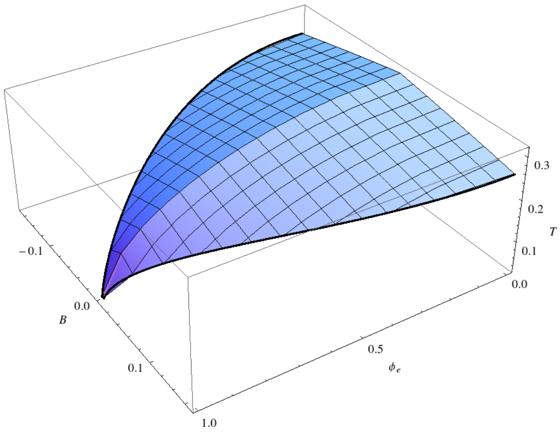


Figure A.2: Black Curve is the Critical Curve in $T - B - \Phi_E$ plane. The manifold is the coexistence plane. Above this plane lies the Small BH, and below it Large BH.

as function of temperature. We write the free energy from equation (3.4):

$$W = M - TS - \Phi_E q_E, \tag{A.13}$$

$$= -\frac{1}{2}\left(\frac{B}{\mathcal{M}}\right)(1-\Phi_E^2)b^4 - \frac{B\mathcal{M}}{2} - \frac{b^{10}}{2}\left(\frac{B}{\mathcal{M}}\right)^3 - \pi T b^8\left(\frac{B}{\mathcal{M}}\right)^2. \tag{A.14}$$

Using positive definiteness of r_+ we can write

$$W = \frac{1}{2}\left|\frac{B}{\mathcal{M}}\right|(1-\Phi_E^2)b^4 + \frac{|B\mathcal{M}|}{2} + \frac{b^{10}}{2}\left|\frac{B}{\mathcal{M}}\right|^3 - \pi T b^8\left|\frac{B}{\mathcal{M}}\right|^2. \tag{A.15}$$

$\partial W/\partial\mathcal{M} = 0$ will give us the back the equation of state (equation 3.5), and we will recover figure (A.1a) and (A.1b). To find the stable solutions of $\mathcal{M}$ in $B \to 0$ limit, we study the extremas of W using (A.14). For $r_+ = -b^4\frac{B}{\mathcal{M}}\Big|_{B\to 0} = 0$ we will have:

$$\mathcal{M} = \pm b^2\sqrt{(1-\Phi_E^2)}. \tag{A.16}$$

Whereas for $r_+ \neq 0$ the only solution is:

$$\mathcal{M} = 0 \quad \text{if} \quad T \geq T_{N(M)} = \frac{\sqrt{3}}{2\pi b}\sqrt{(1-\Phi_E^2)}. \tag{A.17}$$

First one is the background AdS solution which has some net finite magnetization even without any magnetic field. The second solution however corresponds to large black hole. The bound on the temperature is nothing but the maximum nucleation temperature (equation A.11), above which (for $B = 0$) there will always exist large black hole solution with radius

$$r_{+(LBH)} = \frac{2\pi b^2}{3}\left(T + \sqrt{T^2 - T^2_{N(M)}}\right). \tag{A.18}$$

To argue the dominance of the above solutions, we express W in the convenient form of r_+ in $B \to 0$.

$$W = \frac{r_+}{2b^4}\left[(1-\Phi_E^2)b^4 + \mathcal{M}^2 + b^2 r_+^2 - 2\pi T b^4 r_+\right]. \tag{A.19}$$

For AdS solution, $\mathcal{M} = \pm\sqrt{1-\Phi_E^2}$ and $r_+ = 0$, so the free energy vanishes. For the large black hole solution the free energy is given by

$$W_0 = \frac{1}{2}r_{+(LBH)}\left[(1-\Phi_E^2) + \frac{r^2_{+(LBH)}}{b^2} - 2\pi T r_{+(LBH)}\right]. \tag{A.20}$$

When large black hole solution dominates over the global AdS then $W < 0$. Using equation (A.18) which boils down to:

$$T > T_{o(M)} = \frac{1}{\pi b}\sqrt{1-\Phi_E^2}. \tag{A.21}$$

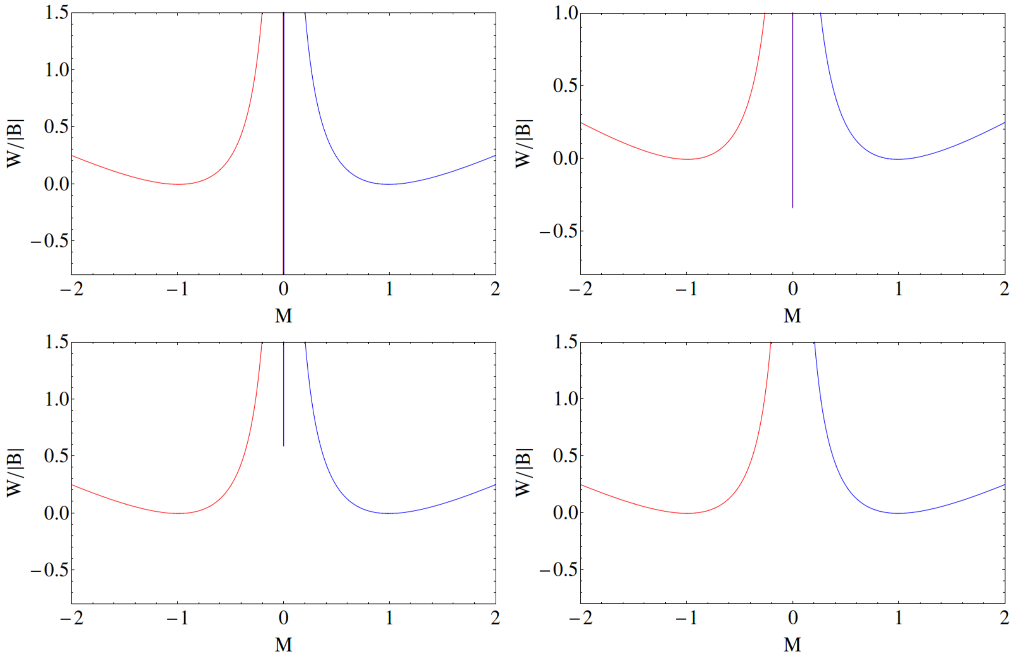
This bound is the maximum coexistence temperature (equation A.12), the temperature at which AdS and large black hole coexist at $B = 0$. Above $T_{o(M)}$, $\mathcal{M} = 0$ solution dominates, while below it $\mathcal{M} = \pm\sqrt{1-\Phi_E^2}$ dominates. But the latter two have the same free energy (equation A.20). Thus system has to spontaneously choose between these two solutions. As soon as B is applied one of the solutions disappears[1], and system has a preferred direction. This behavior has similarity with ferromagnetic system.

Notice that the free energy is always zero in $B \to 0$ limit (except for $\mathcal{M} = 0$). However we can have some insight if we plot $W/|B|$ instead, which keeps the extremas of W unaltered[2] (figure A.3).

Ideally one would be tempted to Taylor expand equation (A.15) near $\mathcal{M} = 0$, and construct a Landau-type model to see the symmetry breaking explicitly. However this is not feasible in our case, as $W/|B|$ is discontinuous and diverging at $\mathcal{M} = 0$ due to the inclusion of $|\mathcal{M}|$, and thus the Taylor expansion breaks down.

[1]Since B and $\mathcal{M}$ must have opposite signs, one of the solutions of $\mathcal{M}$ vanishes depending on the sign of B.

[2]We cannot divide by B as its sign has a dependence on $\mathcal{M}$ due to positivity of r_+



(a) **(b)**

Figure A.3: $W/|B| - \mathcal{M}$ graph for varying T ($\Phi_E = 0.1$) as $B \to 0$. Red curve is for $B \to 0^+$ while Blue is for $B \to 0^-$. In the first and second graph, $T > T_{o(M)}$ (coexistence temperature) and we see that the thermodynamics is governed by $\mathcal{M} = 0$ (Large BH) solution. As T is dropped below T_o (third graph), system spontaneously chooses between the two available solutions of $\mathcal{M} = \pm\sqrt{1 - \Phi_E^2}$ (Small BH or background AdS) as both are equally likely. Further when T is decreased even below $T_{N(M)}$, the $\mathcal{M} = 0$ solution vanishes altogether (fourth graph).

A.4 Details of Diamagnetic and Paramagnetic Phases

Equation (3.8) gives behaviour of χ with B. Thus if we study χ at a constant magnetic field, system will be diamagnetic if:

$$3b^{10}B^2 + 3\mathcal{M}^4 - b^4\mathcal{M}^2\left(1 - \Phi_E^2\right) < 0 \tag{A.22}$$

or

$$3b^{10}B^2 + \mathcal{M}^4 - b^4\mathcal{M}^2\left(1 - \Phi_E^2\right) > 0 \tag{A.23}$$

and otherwise paramagnetic. That is diamagnetic solution is given by:

$$\mathcal{M} < \frac{b^4}{2}\left(\left(1-\Phi_E^2\right) - \sqrt{\left(1-\Phi_E^2\right)^2 - 12b^2B^2}\right) \tag{A.24}$$

$$\begin{aligned}\frac{b^4}{6}\left(\left(1-\Phi_E^2\right) - \sqrt{\left(1-\Phi_E^2\right)^2 - 36b^2B^2}\right) &< \mathcal{M} \\ &< \frac{b^4}{6}\left(\left(1-\Phi_E^2\right) + \sqrt{\left(1-\Phi_E^2\right)^2 - 36b^2B^2}\right)\end{aligned} \tag{A.25}$$

$$\mathcal{M} > \frac{b^4}{2}\left(\left(1-\Phi_E^2\right) + \sqrt{\left(1-\Phi_E^2\right)^2 - 12b^2B^2}\right). \tag{A.26}$$

The middle one is however the unstable phase. If,

$$|B| > B^* = \frac{1}{2\sqrt{3}b}\left(1-\Phi_E^2\right) \tag{A.27}$$

the system will always be diamagnetic regardless of temperature (figure 1.3a). However if,

$$B^* > |B| > |B_c| \tag{A.28}$$

the system will have three phases: diamagnetic, paramagnetic and another diamagnetic (figure 1.3b).

If the magnetic field is even below B_c, system shows the full feature of 5 phases. But as we go below the critical point, we know that there comes an unstable branch which is omitted by the Maxwell's equal area law (figure 3.1). While we do so, a section of the small and large BH is also omitted. As a result below a particular magnetic field $B^{\#}$, the paramagnetic phase of large BH gets cutout and the only stable solution of large BH is diamagnetic phase. $B^{\#}$ will be given by the point where the coexistence magnetic field is same as the $\chi = 0$ point for a $B - \mathcal{M}$ isotherm. For $\Phi_E = 0$ and $b = 1$ we find $B^{\#} \approx 0.154$ whereas $B_c = 0.167$.

B | Non-relativistic Charged Hydrodynamics

B.1 Non-relativistic Electromagnetism

Maxwell's Electrodynamics is a relativistic theory. In fact it was a precursor to Einstein's Special Theory of Relativity. Having a consistent relativistic description of electrodynamics, eradicated any need for a 'non-relativistic' theory of electrodynamics. However as in current context, one needs a Galilean description of electrodynamics, just to keep it consistent with other non-relativistic theories.

Recently in [22], authors discuss the non-relativistic limit of electrodynamics in two distinct ways. Depending on the strength of fields[1], we can have two discrete non-relativistic limits: $|\vec{E}| \gg c|\vec{B}|$ electric limit and $|\vec{E}| \ll c|\vec{B}|$ magnetic limit. We find that the light-cone reduction of relativistic sourceless Maxwell's equations, under certain identifications, gives magnetic limit of Maxwell's equations, which will be our interest of discussion here. A more thorough discussion of non-relativistic electrodynamics can be found in [22].

In arbitrary dimensions, inhomogeneous Maxwell's equations are given by:

$$\nabla_i F^{i0} = -\mu_o c\rho, \qquad \nabla_0 F^{0i} + \nabla_j F^{ji} = -\mu_o J^i, \tag{B.1}$$

while the homogeneous ones (Bianchi identities) are:

$$\epsilon^{\alpha\beta\cdots\mu\nu\sigma}\nabla_\mu F_{\nu\sigma} = 0, \tag{B.2}$$

where $\epsilon^{\alpha\beta\cdots\mu\nu\sigma}$ is the full-rank Levi-Cevita tensor. The last identity follows directly from the gauge invariant form of $F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu$ or the constituent fields[2]: $E^i = -\nabla^i\phi - \nabla_0 A^i$ and $F^{ij} = \nabla^i A^j - \nabla^j A^i$. As LCR gives a non-relativistic theory with gauge invariance, we are interested in a non-relativistic limit which preserves

[1] In this section we use the conventional notation for electrodynamics: E for electric field, B for magnetic field, ρ for charge density, J for charge current etc.

[2] Instead of the conventional magnetic field, we use its 2nd rank dual in our work, as in arbitrary dimensions we do not have electromagnetic duality and magnetic field does not have any fixed rank, while its dual has.

Eqn. (B.2). We will see that the magnetic limit essentially does the same. However there exists a consistent electric limit as well, where Bianchi identities are broken, but inhomogeneous Maxwell's equations are preserved.

In terms of dimensionless fields and sources one can write the inhomogeneous Maxwell's equations as:

$$\alpha \nabla_i E^i = \rho, \qquad \beta \frac{\partial}{\partial t} E^i + \nabla_j F^{ji} = -\frac{\beta}{\alpha} J^i, \tag{B.3}$$

and the Bianchi identities as:

$$\epsilon_{lm\cdots ij} \nabla^i E^j = -\beta \frac{1}{2} \epsilon_{lm\cdots ij} \frac{\partial}{\partial t} F^{ij}, \qquad \epsilon^{m\cdots kij} \nabla_k F_{ij} = 0 \tag{B.4}$$

where α and β are some dimensionless constants given by:

$$\alpha = \frac{[E]\epsilon_o}{[\rho][L]}, \qquad \beta = \frac{[L]}{[T]c}. \tag{B.5}$$

From here we can easily read out, that in non-relativistic limit $\beta \ll 0$. To preserve the Bianchi identities therefore, E^i should be one order smaller than F^{ij}; which is why this limit is called 'magnetic limit'. If we measure smallness in terms of a parameter $\epsilon \sim 1/c$, we will have $E^i \sim \epsilon^{n+1}$ and $F^{ij} \sim \epsilon^n$. Therefore from Eqn. (B.3), first nontrivial order of $\rho \sim \alpha\epsilon^{n+1}$, and $J^i \sim \alpha\epsilon^{n-1}$, and inhomogeneous Maxwell's Equations (in conventional units) reduce to:

$$\nabla_i E^i = \frac{\rho}{\epsilon_o}, \qquad \nabla_j F^{ji} = -\mu_o J^i. \tag{B.6}$$

Essentially we have just dropped the displacement current term from the Ampere's Law. To explicitly see if this limit is Galilean invariant, one can lookup [22]. These are the very equations that have been used in Section (8.1) and (8.5).

One can now push the expansion of Eqn. (B.3) to ϵ^{n+2} order and derive the continuity equation:

$$\frac{\partial}{\partial t}\rho + \nabla_i J^i = 0. \tag{B.7}$$

It should be remembered however that there are mixed orders in this equation and to the highest order it just states: $\nabla_i J^i = 0$, which also follows from Eqn. (B.6). Continuity equation takes its usual form only when first two leading orders of J^i vanish and $\rho \sim J^i$, but in which case one of the Maxwell's equations modifies to: $\nabla_j F^{ji} = 0$. This in conjugation with the Bianchi identity would mean that the magnetic field is constant over space.

Finally in conventional units this limit can be summarized as: $F^{ij} \sim E^i \sim c^{-n}$, and to the maximum order $\rho \sim c^{-n}\epsilon_o$, $J^i \sim c^{-n+2}\epsilon_o$.

B.2 1/c Expansion of Relativistic Fluid Dynamics

In this section we discuss briefly the $1/c$ expansion limit of relativistic fluid dynamics to get a non-relativistic theory, using the prescription of [26]. We consider here just a parity-even fluid for comparison with our work. Parity-odd sector in our case and in [26] are in different hydrodynamic frames and thus are incomparable.

The constitutive equations of a relativistic fluid (with appropriate factors of c) are:

$$\nabla_\mu T^{\mu\nu} = cF_I^{\nu\alpha} J_{I\alpha}, \qquad \nabla_\mu J_I^{a\mu} = 0, \tag{B.8}$$

where

$$T^{\mu\nu} = (E+P)u^\mu u^\nu + Pg^{\mu\nu} + \Pi^{\mu\nu}, \qquad J_I^{a\mu} = Q_I u^\mu + \Upsilon_I^\mu. \tag{B.9}$$

Respective dissipative terms are given as:

$$\Pi^{\mu\nu} = -2\eta\tau^{\mu\nu} - \zeta\theta P^{\mu\nu}, \qquad \Upsilon_I^\mu = -\frac{1}{c^2}T\lambda_{IJ}P^{\mu\nu}\nabla_\nu\left(\frac{M_J}{T}\right) + \frac{1}{c}\lambda_{IJ}E_J^\mu. \tag{B.10}$$

We can expand E and M_I in terms of their potential and kinetic parts:

$$E = Rc^2 + \mathcal{E}, \qquad M_I = \frac{1}{\mathcal{K}}m_I c^2 + \mathcal{M}_I. \tag{B.11}$$

Here m_I is the mass is to 'I'th charge ratio of constituent particles in their local rest frame, which is assumed to be constant. $\mathcal{K}$ is the total number of $U(1)$ charges. Non-relativistic mass and energy density are related to their relativistic counterparts as:

$$\rho = R\Gamma, \qquad \epsilon = \mathcal{E}\Gamma \tag{B.12}$$

where $\Gamma = (1-\mathbf{v}^2/c^2)^{-1}$. Since the fluid under consideration is single component, $R/Q_I = m_I$ is a constant. Non-relativistic charge density is defined as:

$$q_I = J_I^0. \tag{B.13}$$

Pressure and Temperature are however kept same, which to be consistent with the main text notation we will denote as: $\tau = T$ and $p = P$.

Before continuing with expansion we need to fix the order of various quantities. $\rho, \epsilon, p, \tau, v^i$ can be thought of finite order without much ambiguity. m_I on the other hand is quite sensitive to the kind of system under consideration. Let us consider a charged fluid made of 'ions' where $m_I \sim c^2$; m_I cannot be too low, or else the fluid would start coupling to the background fields. Correspondingly the charge density q_I would be of order c^{-2}. Further, to keep the thermodynamics intact, one has to assume m_I to be of order of c^2. Finally, for external fields to have finite effect (e.g. force) on the fluid, $\epsilon_I^i \sim \beta_I^{ij} \sim c^2$.

Using this information we can reduce the energy-momentum tensor to:

$$
\begin{aligned}
T^{00} &= \rho c^2 + \frac{1}{2}\rho \mathbf{v}^2 + \epsilon + \mathcal{O}(1/c) \\
T^{i0} &= \rho v^i c + \left(\frac{1}{2}\rho \mathbf{v^2} + \epsilon + p\right)\frac{v^i}{c} + \frac{1}{c}\pi^{ij} v_j + \mathcal{O}(1/c^2) \\
T^{ij} &= t^{ij} + \mathcal{O}(1/c), \qquad t^{ij} = \rho v^i v^j + p g^{ij} + \pi^{ij}
\end{aligned} \tag{B.14}
$$

where we have used:

$$
n = \frac{\eta}{c}, \qquad z = \frac{\zeta}{c}, \tag{B.15}
$$

$$
\pi^{ij} = -n\left(\nabla^i v^j + \nabla^j v^i - g^{ij}\frac{2}{d}\nabla_k v^k\right) - z g^{ij}\nabla_k v^k \tag{B.16}
$$

Energy-momentum conservation equations, at highest order, will then reduce to:

$$
\partial_t \rho + \partial_i(\rho v^i) = 0, \qquad \partial_t(\rho v^i) + \partial_i t^{ij} = \epsilon_I^i q_I + \beta_I^{ij} j_{Ij}. \tag{B.17}
$$

It will be worth to mention here the underlying assumption: $n, z \sim 1$ which is just a observational fact. This is precisely the reason why no dissipative corrections appear to the continuity equation. We can now expand the charge current:

$$
\begin{aligned}
J_I^0 &= q_I, \qquad q_I = Q_I \Gamma + \mathcal{O}(1/c^6) \\
J_I^i &= \frac{1}{c} j_I^i + \mathcal{O}(1/c^7), \qquad j_I^i = q_I v^i + \varsigma_I^i
\end{aligned} \tag{B.18}
$$

where we have used:

$$
\tilde{\sigma}_{IJ} = \frac{\lambda_{IJ}}{c}, \qquad \tilde{\gamma}_{IJ} = \frac{\gamma_{IJ}}{c} \tag{B.19}
$$

$$
\varsigma_I^i = c^2 \tilde{\sigma}_{IJ} \frac{m_J}{\mathcal{K}\tau} \nabla^i \tau \tag{B.20}
$$

Here again we have used a physical input to fix the order of $\tilde{\sigma}_{IJ}$. We would demand that in the non-relativistic theory, charge to mass ratio should be a constant. Therefore charge continuity and mass continuity equations should be the same upto leading order in $1/c$, and hence ς_I^i should at maximum be of the order of c^{-4}. It further implies that $\tilde{\sigma}_{IJ} \sim c^{-8}$. As a consequence, effects like electric conductivity are suppressed in whole theory.

Finally we use: $\nabla_\mu \left(T^{\mu 0} - \frac{1}{\mathcal{K}} m_I c^2 J_I^\mu\right) = 0$ to get the energy conservation:

$$\partial_t \left[\frac{1}{2}\rho \mathbf{v}^2 + \epsilon\right] + \partial_i \left[v^i \left(\frac{1}{2}\rho \mathbf{v^2} + \epsilon + P\right) + \varsigma^i\right] = 0, \tag{B.21}$$

$$\varsigma^i = \pi^{ij} v_j - \kappa \nabla^i \tau, \tag{B.22}$$

where we have identified the thermal conductivity:

$$\kappa = \tilde{\sigma}_{IJ} \frac{c^4}{\mathcal{K}^2} \frac{m_I^a m_J^a}{\tau}. \tag{B.23}$$

which is of finite order.

The non-relativistic theory has essentially turned out to be identical to the chargeless fluid discussed in [20], because for uni-component fluids, charge density serves just as number density upto a factor. For multi-particle fluids however this might become more interesting, as then one does not have to expect charge is to mass ratio to be constant. In fact one can have some chargeless particles also in the system.

B.2.1 1/c Expansion of Thermodynamics

First law of thermodynamics and the Euler's relation in the relativistic theory are given by:

$$\mathrm{d}E = T\mathrm{d}S + M_I \mathrm{d}Q_I, \qquad E + P = TS + M_I Q_I. \tag{B.24}$$

Under $1/c$ expansion they reduce to:

$$\mathrm{d}\epsilon = \epsilon \mathrm{d}s + \mu_I \mathrm{d}q_I, \qquad \epsilon + p = \tau s + \mu_I q_I, \tag{B.25}$$

which are just the non-relativistic analogues of the same equations. We therefore conclude that the non-relativistic system gained by $1/c$ expansion follows extensivity.

Now lets have a look at the second law of thermodynamics in the relativistic side, which mentions: $\nabla_\mu J^\mu_S \geq 0$. From Eqn. (7.18) we know the positive definite form of entropy current:

$$\nabla_\mu J^\mu_S = \frac{1}{T}\eta\tau^{\mu\nu}\tau_{\mu\nu}+\frac{1}{T}\zeta\theta^2+\left[\frac{E^\alpha_I}{c^2T}-\frac{1}{c}P^{\alpha\mu}\nabla_\mu\left(\frac{M_I}{T}\right)\right]\varrho_{IJ}\left[\frac{E_{J\alpha}}{c^2T}-\frac{1}{c}P_{\alpha\nu}\nabla^\nu\left(\frac{M_J}{T}\right)\right]. \tag{B.26}$$

In highest order this equation for non-relativistic systems says that:

$$\nabla_t s + \nabla_i j^i_S = \frac{1}{2\tau}n\sigma^{ij}\sigma_{ij} + \frac{1}{\tau}z\left(\nabla_k v^k\right) + \frac{\kappa}{\tau^2}\left(\nabla^i\tau\right)^2. \tag{B.27}$$

But if we look at the respective coefficients from the relativistic side:

$$n = \frac{\eta}{c} \geq 0, \qquad z = \frac{\zeta}{c} \geq 0, \qquad \kappa = \varrho_{IJ}\frac{c^3}{\mathcal{K}^2}\frac{m^a_I m^a_J}{T^2} \geq 0. \tag{B.28}$$

We hence see that the relativistic entropy positivity implies non-relativistic entropy positivity.

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